Axiomatizability criteria in modal logic

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Abstract model theory

Consider:

- $\mathcal{L}$ — a language (any set; its elements are called formulas)
- $\mathcal{S}$ — a class of structures or models
- $\models$ — a truth relation: $M \models A$ between $M \in \mathcal{S}$ and $A \in \mathcal{L}$

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How can we "characterize"

- classes of models from $\mathcal{S}$ definable by a single formula from $\mathcal{L}$?
- classes of models from $\mathcal{S}$ definable by a set of formulas from $\mathcal{L}$?

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For a set of formulas $\Gamma \subseteq \mathcal{L}$ and a class of models $\mathbb{K} \subseteq \mathcal{S}$, we denote:

$$\begin{aligned} \mathrm{Models}(\Gamma) &:= \{ M \in \mathcal{S} \mid M \models \Gamma \} \\ \mathrm{Theory}(\mathbb{K}) &:= \{ A \in \mathcal{L} \mid \mathbb{K} \models A \} \end{aligned}$$

The 4 "species" of classes

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$\mathbb{K} \in \Cap\mathbb{L}$ if $\mathbb{K} = \mathsf{Models}(\Gamma)$, for some set of formulas $\Gamma \subseteq \mathcal{L}$.

Equivalently: if $\mathbb{K} = \bigcap_{i \in I} \mathbb{K}_i$ for some classes $\mathbb{K}_i \in \mathbb{L}$.

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$\mathbb{K} \in \Cup\Cap\mathbb{L}$	if	$\mathbb{K} = \bigcup\limits_{i \in I} \bigcap\limits_{j \in J_i} \mathbb{K}_{i,j}$ for some classes $\mathbb{K}_{i,j} \in \mathbb{L}$.

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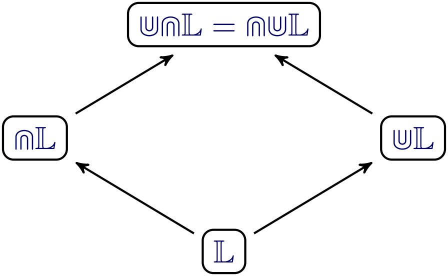
For the "elementary" (i.e. first-order) language $\mathcal{L}$, the terminology is:

$\mathbb{K} \in \mathbb{L}$	— an	elementary class of models (*finitely axiomatizable*)
$\mathbb{K} \in \Cap\mathbb{L}$	— a	Δ-elementary class of models (*axiomatizable*)
$\mathbb{K} \in \Cup\mathbb{L}$	— a	Σ-elementary class of models (*co-axiomatizable?*)
$\mathbb{K} \in \Cup\Cap\mathbb{L}$	— a	$\Sigma\Delta$-elementary class of models

The hierarchy of the 4 species of classes



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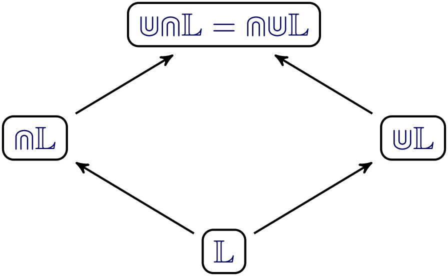

- Classes in $\mathbb{L}$: the classes of all groups, all rings, all fields
- Classes in $\Cap\mathbb{L}$: infinite groups, infinite rings, infinite fields
- Classes in $\Cup\mathbb{L}$: finite groups, finite rings, finite fields
- Classes in $\Cup\Cap\mathbb{L}$: infinite fields of characteristic $p > 0$;
  infinite finitely dimensional vector spaces
- Not even in $\Cup\Cap\mathbb{L}$: well-ordered sets, periodic groups, simple groups

Isomorphism of two models

$M \cong N \quad \leftrightharpoons \quad \exists$ bijection that preserves all predicates and functions

First-order language: Relations / functions between models

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Ultraproduct of a family of models: $M = \prod_{i \in I}^{U} M_i$

Łós' Theorem: $\quad M \models A \quad \iff \quad \{\, i \in I \mid M_i \models A \,\} \in U$

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Ultrapower of a model N

If every $M_i = N$ then their ultraproduct is called the ultrapower: $M = N^U$

A model and its ultrapower are elementary equivalent: $N \equiv_{\mathrm{FO}} N^U$

Theorem (Keisler, 1961)

	Both	$\mathbb{K}$	$\overline{\mathbb{K}}$
$\mathbb{K} \in \Cup\Cap\mathbb{L}$	$\equiv_{\mathsf{FO}}$		
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Legend: УП = ultraproduct

First-order language | Criteria for the 4 species

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(Keisler, 1961; Shelah, 1971)

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Main reason for the symmetry in the above tables:

$$M \not\models A \iff M \models \neg A$$

Modal language | Kripke semantics

Formulas: $p_i \mid \neg A \mid (A \wedge B) \mid (A \vee B) \mid (A \to B) \mid \Box A$

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Kripke semantics:

Kripke model: $M = (W, R, V)$, where

- $W \neq \varnothing$ — a nonempty set of worlds
- $R \subseteq W \times W$ — a accessibility relation between worlds
- $V(p_i) \subseteq W$ — a valuation of variables

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Truth of a formula is defined in a pointed model (M, x):

$$
\begin{array}{lcl}
M, x \models p_i & \leftrightharpoons & x \in V(p_i) \\
M, x \models \neg A & \leftrightharpoons & M, x \not\models A \\
M, x \models A \wedge B & \leftrightharpoons & M, x \models A \quad \text{and} \quad M, x \models B \\
M, x \models A \vee B & \leftrightharpoons & M, x \models A \quad \text{or} \quad M, x \models B \\
M, x \models A \to B & \leftrightharpoons & M, x \models A \quad \Rightarrow \quad M, x \models B \\
M, x \models \Box A & \leftrightharpoons & \text{for every } y \in W \ (x R y \ \Rightarrow \ M, y \models A)
\end{array}
$$

Truth of a formula in a model: $M \models A$ if $\forall x \in W \ M, x \models A$.

Modal equivalence of two (pointed) Kripke models

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$M, a \simeq N, b$ — respects the valuation of variables
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Global bisimulation between Kripke models

$M :\simeq: N$ — bisimulation that covers the whole models M and N

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Generated submodel: $M \hookrightarrow N$
Disjoint union of models: $M = \biguplus_{i \in I} M_i$

Theorem: for pointed Kripke models (Maarten de Rijke, 1993)

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$\mathbb{K} \in \mathbb{U}\mathbb{\cap}\mathbb{L}$	$\equiv_{\mathsf{ML}}$		
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Modal language: "purely modal" operations on models

Ultra-extension of a Kripke model $M = (W, R, V)$

— is a Kripke model $M^{\mathfrak{ue}} = (W^{\mathfrak{ue}}, R^{\mathfrak{ue}}, V^{\mathfrak{ue}})$, where

worlds:	$W^{\mathfrak{ue}}$	—	all ultrafilters over the set W;
relation:	$\alpha\, R^{\mathfrak{ue}}\, \beta$	$\leftrightharpoons$	$\forall X \subseteq W\ (\Diamond X \in \alpha \ \Leftarrow\ X \in \beta)$
		$\Leftrightarrow$	$\forall X \subseteq W\ (\Box X \in \alpha \ \Rightarrow\ X \in \beta)$
valuation:	$\alpha \models p_i$	$\leftrightharpoons$	$V(p_i) \in \alpha$

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A model and its ultra-extension are modally equivalent: $M \equiv_{\mathsf{ML}} M^{\mathfrak{ue}}$

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Ultra-union of a family of pointed Kripke models $(M_i, a_i)_{i\in I}$

$M = \Big((\biguplus_{i\in I} M_i)^{\mathfrak{ue}},\ \alpha \Big)$, all co-finite subsets of $\{\, \langle a_i, i\rangle \mid i \in I \,\}$ are in α.

Observation. Ultra-union behaves like the ultra-product.

Modal language: "purely modal" criteria

Theorem: for pointed Kripke models (Yde Venema, 1999; E.Z. 2017)

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?		
$:\simeq:$	$\hookrightarrow\ \uplus\ \mathfrak{ue}$	$\mathfrak{ue}$
?		

Universal modality | "purely modal" criteria

Theorem: for pointed Kripke models (possibly known; E.Z. 2017)

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$\mathbb{K} \in \Cup\Cap\mathbb{L}$	$\equiv_{\mathsf{ML}\forall}$		
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Further directions

- Criteria for other semantics of the modal language:
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  - add modalities: converse (tense) $\Box^{-1}$, inequality $[\neq]$, transitive closure $\boxplus$, graded modalities $\Diamond^{\geqslant n}$, hybrid logic (nominals) $@_i$
  - infinitary modal language (for any set Φ of formulas $\bigwedge\Phi$ is a formula):
    – classes of models definable by a single infinitary modal formula,
    – classes of models definable by a class (!) of infinitary modal formula,
  - intuitionistic propositional language
  - modal predicate language

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Thank you!

The modality of inequality $[\neq]$ | Check!

Theorem: for pointed models (M. de Rijke, 1992; E.Z. 2017)

	Both	$\mathbb{K}$	$\overline{\mathbb{K}}$
$\mathbb{K} \in \Cup\Cap\mathbb{L}$	$\equiv_{\mathsf{ML}\neq}$		
$\mathbb{K} \in \Cup\mathbb{L}$	$\equiv_{\mathsf{ML}\neq}$		УП
$\mathbb{K} \in \Cap\mathbb{L}$	$\equiv_{\mathsf{ML}\neq}$	УП	
$\mathbb{K} \in \mathbb{L}$	$\equiv_{\mathsf{ML}\neq}$	УП	УП

	Both	$\mathbb{K}$	$\overline{\mathbb{K}}$
$\mathbb{K} \in \Cup\Cap\mathbb{L}$	$\simeq_{\neq}$	УС	УС
$\mathbb{K} \in \Cup\mathbb{L}$	$\simeq_{\neq}$	УС	УП
$\mathbb{K} \in \Cap\mathbb{L}$	$\simeq_{\neq}$	УП	УС
$\mathbb{K} \in \mathbb{L}$	$\simeq_{\neq}$	УП	УП

Theorem: for models (M. de Rijke, 1992; E.Z. 2017)

	Both	$\mathbb{K}$	$\overline{\mathbb{K}}$
$\mathbb{K} \in \Cup\Cap\mathbb{L}$	$\equiv_{\mathsf{ML}\neq}$	$\hookrightarrow$	
$\mathbb{K} \in \Cup\mathbb{L}$	$\equiv_{\mathsf{ML}\neq}$	$\hookrightarrow$	УП
$\mathbb{K} \in \Cap\mathbb{L}$	$\equiv_{\mathsf{ML}\neq}$	$\hookrightarrow$ $\uplus$ УП	
$\mathbb{K} \in \mathbb{L}$	$\equiv_{\mathsf{ML}\neq}$	$\hookrightarrow$ $\uplus$ УП	УП

Both	$\mathbb{K}$	$\overline{\mathbb{K}}$
$:\simeq:_{\neq}$	$\hookrightarrow$ УС	УС
$:\simeq:_{\neq}$	$\hookrightarrow$ УС	УП
$:\simeq:_{\neq}$	$\hookrightarrow$ $\uplus$ УП	УС
$:\simeq:_{\neq}$	$\hookrightarrow$ $\uplus$ УП	УП

Tense language | Criteria (check!)

Theorem: for pointed models (who? E.Z. 2017)

	Both	$\mathbb{K}$	$\overline{\mathbb{K}}$
$\mathbb{K} \in \Cup\Cap\mathbb{L}$	$\equiv_{\mathsf{ML.t}}$		
$\mathbb{K} \in \Cup\mathbb{L}$	$\equiv_{\mathsf{ML.t}}$		УП
$\mathbb{K} \in \Cap\mathbb{L}$	$\equiv_{\mathsf{ML.t}}$	УП	
$\mathbb{K} \in \mathbb{L}$	$\equiv_{\mathsf{ML.t}}$	УП	УП

	Both	$\mathbb{K}$	$\overline{\mathbb{K}}$
$\mathbb{K} \in \Cup\Cap\mathbb{L}$	$\simeq_t$	УС	УС
$\mathbb{K} \in \Cup\mathbb{L}$	$\simeq_t$	УС	УП
$\mathbb{K} \in \Cap\mathbb{L}$	$\simeq_t$	УП	УС
$\mathbb{K} \in \mathbb{L}$	$\simeq_t$	УП	УП

Theorem: for models (who?; E.Z. 2017)

	Both	$\mathbb{K}$	$\overline{\mathbb{K}}$
$\mathbb{K} \in \Cup\Cap\mathbb{L}$	$\equiv_{\mathsf{ML.t}}$	$\hookrightarrow$	
$\mathbb{K} \in \Cup\mathbb{L}$	$\equiv_{\mathsf{ML.t}}$	$\hookrightarrow$	УП
$\mathbb{K} \in \Cap\mathbb{L}$	$\equiv_{\mathsf{ML.t}}$	$\hookrightarrow$ $\uplus$ УП	
$\mathbb{K} \in \mathbb{L}$	$\equiv_{\mathsf{ML.t}}$	$\hookrightarrow$ $\uplus$ УП	УП

Both	$\mathbb{K}$	$\overline{\mathbb{K}}$
$:\simeq:_t$	$\hookrightarrow_t$ УС	УС
$:\simeq:_t$	$\hookrightarrow_t$ УС	УП
$:\simeq:_t$	$\hookrightarrow_t$ $\uplus$ УП	УС
$:\simeq:_t$	$\hookrightarrow_t$ $\uplus$ УП	УП

Graded modalities $\Diamond^{\geqslant n}$ | Criteria

Theorem: for pointed models (Maarten de Rijke, 2000)

	Both	$\mathbb{K}$	$\overline{\mathbb{K}}$
$\mathbb{K} \in \mathbb{U}\Cap\mathbb{L}$	$\equiv_{MLG}$		
$\mathbb{K} \in \mathbb{U}\mathbb{L}$	$\equiv_{MLG}$		УП
$\mathbb{K} \in \Cap\mathbb{L}$	$\equiv_{MLG}$	УП	
$\mathbb{K} \in \mathbb{L}$	$\equiv_{MLG}$	УП	УП

	Both	$\mathbb{K}$	$\overline{\mathbb{K}}$
$\mathbb{K} \in \mathbb{U}\Cap\mathbb{L}$	$\simeq_G$	УС	УС
$\mathbb{K} \in \mathbb{U}\mathbb{L}$	$\simeq_G$	УС	УП
$\mathbb{K} \in \Cap\mathbb{L}$	$\simeq_G$	УП	УС
$\mathbb{K} \in \mathbb{L}$	$\simeq_G$	УП	УП

Theorem: for models (Maarten de Rijke did not write, check!)

	Both	$\mathbb{K}$	$\overline{\mathbb{K}}$
$\mathbb{K} \in \mathbb{U}\Cap\mathbb{L}$	$\equiv_{MLG}$	$\hookrightarrow$	
$\mathbb{K} \in \mathbb{U}\mathbb{L}$	$\equiv_{MLG}$	$\hookrightarrow$	УП
$\mathbb{K} \in \Cap\mathbb{L}$	$\equiv_{MLG}$	$\hookrightarrow$ $\uplus$ УП	
$\mathbb{K} \in \mathbb{L}$	$\equiv_{MLG}$	$\hookrightarrow$ $\uplus$ УП	УП

Both	$\mathbb{K}$	$\overline{\mathbb{K}}$
$:\simeq:_G$	$\hookrightarrow$ УС	УС
$:\simeq:_G$	$\hookrightarrow$ УС	УП
$:\simeq:_G$	$\hookrightarrow$ $\uplus$ УП	УС
$:\simeq:_G$	$\hookrightarrow$ $\uplus$ УП	УП

Theorem: for pointed models (Piet Rodenburg 1986)

	Both	$\mathbb{K}$	$\overline{\mathbb{K}}$
$\mathbb{K} \in \Cup\Cap\mathbb{L}$			
$\mathbb{K} \in \Cup\mathbb{L}$			
$\mathbb{K} \in \Cap\mathbb{L}$			
$\mathbb{K} \in \mathbb{L}$			

	Both	$\mathbb{K}$	$\overline{\mathbb{K}}$
$\mathbb{K} \in \Cup\Cap\mathbb{L}$			
$\mathbb{K} \in \Cup\mathbb{L}$			
$\mathbb{K} \in \Cap\mathbb{L}$			
$\mathbb{K} \in \mathbb{L}$			

Theorem: for models (Piet Rodenburg 1986)

	Both	$\mathbb{K}$	$\overline{\mathbb{K}}$
$\mathbb{K} \in \Cup\Cap\mathbb{L}$			
$\mathbb{K} \in \Cup\mathbb{L}$			
$\mathbb{K} \in \Cap\mathbb{L}$			
$\mathbb{K} \in \mathbb{L}$			

Both	$\mathbb{K}$	$\overline{\mathbb{K}}$
$:\simeq:$	$\hookrightarrow$ УС	УС
$:\simeq:$	$\hookrightarrow$ УС	УП
$:\simeq:$	$\hookrightarrow$ $\uplus$ УП	УС
$:\simeq:$	$\hookrightarrow$ $\uplus$ УП	УП

Theorem: for pointed models

	Both	$\mathbb{K}$	$\overline{\mathbb{K}}$
$\mathbb{K} \in \mathbb{U}\mathbb{\cap}\mathbb{L}$			
$\mathbb{K} \in \mathbb{U}\mathbb{L}$			
$\mathbb{K} \in \mathbb{\cap}\mathbb{L}$			
$\mathbb{K} \in \mathbb{L}$			

	Both	$\mathbb{K}$	$\overline{\mathbb{K}}$
$\mathbb{K} \in \mathbb{U}\mathbb{\cap}\mathbb{L}$			
$\mathbb{K} \in \mathbb{U}\mathbb{L}$			
$\mathbb{K} \in \mathbb{\cap}\mathbb{L}$			
$\mathbb{K} \in \mathbb{L}$			

Theorem: for models (Robert Goldblatt 2005)

	Both	$\mathbb{K}$	$\overline{\mathbb{K}}$
$\mathbb{K} \in \mathbb{U}\mathbb{\cap}\mathbb{L}$			
$\mathbb{K} \in \mathbb{U}\mathbb{L}$			
$\mathbb{K} \in \mathbb{\cap}\mathbb{L}$			
$\mathbb{K} \in \mathbb{L}$			

Both	$\mathbb{K}$	$\overline{\mathbb{K}}$
$:\simeq:$	$\hookrightarrow$ $\uplus$ $\mathfrak{pe}$	$\mathfrak{pe}$
?		