CONTENTS

1. Modal logic

1.1. Syntax
1.1.1. Language
1.1.2. Axiom systems
1.1.3. Logics
1.1.4. Maximal consistent sets
1.1.5. Theories
1.2. Semantics
1.2.1. Model theoretical semantics
1.2.2. Algebraic semantics
1.2.3. Soundness and completeness
1.2.4. Canonical models
1.2.5. The finite model property
1.2.6 Filtrations
1.2.7. Decidability

2. Ancestral logic

2.1. Semantics
2.1.1. Model theory
2.1.2 Compactness and noncompactness
2.2. Syntax
2.2.1. Axiom systems for ancestral logic
2.2.2. Comparison with tense-logic
2.3. Completeness
2.3.1. Canonical models
2.3.2. Filtrations

3. Dynamic logic

3.1. Semantics
3.1.1. Frames
3.1.2. Language
3.1.3. Models
3.2. Syntax
3.2.1. Axiom systems
3.2.2. Fischer/Ladner closure
3.3. Completeness
3.3.1. Canonical models
3.3.2. Filtrations
3.3.3. Completeness of PDL
3.4. Limitations of PDL
3.4.1. Path semantics
3.4.2. A warning regarding IF - THEN - ELSE and WHILE
3.4.3. The delta operator

4. Background

4.1. Historical remarks
4.1.1. Modal logic
4.1.2. Dynamic logic
4.2. Selective bibliography

1. Modal logic

1.1. Syntax

1. Language

If someone wants a brief description of modal logic, perhaps one might reply that it is ordinary logic with one or two extra so-called modal operators □ and ◊, each taking a formula as an argument and yielding a formula as a result. One might also add that people who do modal logic usually have in mind an informal understanding, however abstract, of those operators, associating with them readings such as "it is necessary (in a certain sense) that" and "it is possible (in a certain sense) that", respectively.

As it turns out, in classical modal logic the two modal operators are interdefinable, so it is customary to posit one of them as primitive (nowadays usually □) and construct the other with the help of definition.

Thus in modal propositional logic we have an *alphabet* consisting of certain *primitive symbols*:

(i) a denumerable supply of propositional letters,
(ii) a finite truth-functionally complete set of Boolean operators,
(iii) the modal operator □,
(iv) grouping devices (for example, parentheses and commas).

From these we build formulæ in the usual way. Every finite string of primitive symbols in an *expression*, and formulæ are expressions of a certain sort. This is by no means problematic, and so it may seem like an act of supererogation to offer a definition. However, in order later to be able to give rigorous proofs by induction on the structure of formulæ we need a precise inductive definition of formula.

1. Every propositional letter is a *formula*.
2. If o is an n-ary Boolean operator and A_0, ..., A_{n-1} are *formulæ*, then o(A_0, ..., A_{n-1}) is also a *formula*.
3. If A is a *formula*, then □A is also a *formula*.
4. Nothing is a *formula* except by virtue of clauses 1-3.

Strict adherence to this definition will produce formulæ in a form related to what is known as Polish notation. However, we will feel free to use customary ways of writing formulæ without further explanation, for example writing $\Box A \supset (B \wedge C)$ rather than $\supset(\Box A, \wedge(B, C))$. We will also omit parentheses in unsystematic ways when we think it can be done without causing confusion.

Enumeration lemma. There is an effective enumeration (without repetitions) of the set Φ of all formulæ.

Proof. The proof will depend on the precise nature of the alphabet, something we have left vague here. Rather than making our theory comprehensive (and losing some comprehension in the process), let us discuss an example. Suppose that $\top$, $\bot$, $\wedge$, $\vee$, $\supset$, $\equiv$, $\neg$ are our Boolean operators. The set of our propositional letters is denumerable, hence there is an enumeration of them, say P_0, P_1, ..., P_n, ...

To make it absolutely clear what is going on we will now introduce a new auxiliary alphabet. Let **P** and ′ be two new symbols. Our propositional letters will be complex symbols $\mathbf{P}^{(n)}$, where (n) stands for a string of n occurrences of ′. Our Boolean operators, corresponding to $\top$, $\bot$, $\wedge$, $\vee$, $\supset$, $\equiv$, and $\neg$, respectively, will be T, F, K, A, C, E, and N. Our only modal operator, corresponding to $\Box$, is L. There are no grouping devices. Thus the formulæ of our auxiliary language are in true Polish notation. Moreover, there is a one-one correspondence between formulæ written in the new language and formulæ written in the old one (the "real" formulæ). For example, corresponding to $\Box P_2 \supset (P_0 \wedge P_3)$ we have CL**P**′′K**PP**′′′.

Define the following correspondence between the numbers less than 10 and the symbols of our auxiliary alphabet:

0:′, 1:**P**, 2:T, 3:F, 4:K, 5:A, 6:C, 7:E, 8:N, 9:L.

Evidently this definition induces a one-one correspondence between expressions in our auxiliary language and natural numbers: given any number written in decimal notation, we can at once read off the corresponding expression in the auxiliary language. Conversely, any expression in that language can be read as a numeral written on base 10.

In order to produce an enumeration of all formulæ of the old language we go through the natural numbers one by one, starting with 0. For each number, we reconstruct the corresponding expression in the auxiliary language. Is it a formula of that langage? That is a question that can be decided in a finite number of steps. If the answer is no, then we proceed to the next number. If the answer is yes, then we add the corresponding old formula to our enumeration. Thus the first ten entries in our enumeration will be P_0, $\top$, $\bot$, P_1, $\neg P_0$, $\Box P_0$, P_2, $P_0 \wedge P_0$, $P_0 \wedge \top$, and $P_0 \wedge \bot$. All entries in the enumeration are formulæ of the original language, and sooner or later every such formula will appear in it —but only once. ■

By assumption, in our language Boolean negation, $\neg$, is available. Hence the following definition makes sense: for every formula A,

$$\Diamond A =_{df} \neg\Box\neg A.$$

At this stage, all that this means is that $\Diamond A$ is a certain string of primitive symbols. Only later will we be able to prove that the definition suits our informal understanding of the modal operators.

2. Axiom systems

An *axiom system* $\mathfrak{S}$ is a pair (Ξ, R) where Ξ is a set of formulæ and R is a set of functions from sets of formulæ to sets of formulæ. In other words, if Φ is the set of all formulæ, then $\Xi \subseteq \Phi$ and, for every $\rho \in R$, $\rho \subseteq \mathfrak{P}\Phi \times \mathfrak{P}\Phi$. The elements of Ξ are called *axioms*, those of R *inference rules*. A rule ρ is *finitary* if $(\Sigma, \Theta) \in \rho$ implies that Σ is finite. (The rules considered in these notes are all finitary.)

If $(\Gamma, \Theta) \in \rho$, for some rule ρ and $A \in \Theta$, then we say that A *follows* from Γ by ρ. By a *formal proof* in $\mathfrak{S}$ we mean a string $A_0, \ldots, A_n$ of formulæ such that, for all $i \le n$, A_i is either an axiom of $\mathfrak{S}$ or else follows from some set $\{A_j : j < i\}$ by some rule of $\mathfrak{S}$. We say that A is a *(formal) theorem* of $\mathfrak{S}$ if, for some $n \ge 0$ and some formulæ $A_0, \ldots, A_n$, $\langle A_0, \ldots, A_n \rangle$ is a formal proof in $\mathfrak{S}$ and $A_n = A$. We write $\Lambda\mathfrak{S}$ for the set of theorems of $\mathfrak{S}$.

A *(formula) schema* is the set of all the substitution instances of some particular formula. Following what is nowadays common practice we will specify the axioms of our axiom systems with the help of axiom schemata. Here are some important examples of such schemata:

(0)	$\top$,
(1)	$\Box\top$,
(2)	$\Box(A \wedge B) \equiv (\Box A \wedge \Box B)$,
(T)	$\Box A \supset A$,
(4)	$\Box A \supset \Box\Box A$,
(5)	$\neg\Box A \supset \Box\neg\Box A$.

Examples of rules are modus ponens (MP) and the rule of congruence (RC):

(MP) $\{A, A \supset B\} \longmapsto \{B\}$,
(RC) $\{A \equiv B\} \longmapsto \{\Box A \equiv \Box B\}$.

Other schemata and rules of interest are:

(K) $\Box(A \supset B) \supset (\Box A \supset \Box B)$,
(D) $\neg(\Box A \wedge \Box\neg A)$,
(B) $A \vee \Box\neg\Box A$,

(RN) $\{A\} \longmapsto \{\Box A\}$,
(RE) $\{A \equiv B, C\} \longmapsto$ {C' is like C except that one occurrence of A in C has been replaced by an occurrence of B},
(RPE) $\{A \equiv B\} \longmapsto$ {$C \equiv C'$: C' is like C except that one occurrence of A in C has been replaced by an occurrence of B},
(RS) $\{(A_0 \wedge ... \wedge A_{n-1}) \supset B\} \longmapsto$ $\{(\Box A_0 \wedge ... \wedge \Box A_{n-1}) \supset \Box B\}$ (for all $n \geq 0$).

To aid memory: K for "Kripke", D for "deontic", B for "Brouwer", RN for the Rule of Necessitation, RPE for the Replacement of Provable (Material) Equivalents, and RS for Scott's Rule.

Yet another important rule is that of uniform substitution, (US). By a *substitution function* we shall mean any function s from the set of propositional letters to the set of all formulæ. Given a substitution function we extend it ("lift it") to a function s* from the set of all formulæ; the range is still the set of all formulæ:

$s^*(P) = sP$, for every propositional letter,
$s^*(o(A_0, ..., A_{n-1})) = o(s^*A_0, ..., s^*A_{n-1})$, if o is any n-ary operator and $A_0, ..., A_{n-1}$ are any n formulæ.

We say that s*(A) is a *substitution instance* of A. Indeed, if $P_0, ..., P_{m-1}$ are all the propositional letters occurring in A, then s*A is the result of substituting sP_0 for P_0, ..., sP_{m-1} for P_{m-1}. The rule (US) can now be stated as follows:

(US) $\{A\} \longmapsto$ {A' : A' is a substitution instance of A}.

3. Logics

A set Σ *provides* a schema if it contains every formula in the schema. Moreover, Σ *provides* a rule ρ (equivalently, Σ is *closed under* ρ) if $B \in \Sigma$ whenever $A_0, \ldots, A_{n-1} \in \Sigma$ and B follows from $\{A_0, \ldots, A_{n-1}\}$ by ρ. A *logic* is a set of formulæ that contains all tautologies and is closed under (MP) and (US). The formulæ in a logic L are called the *theses* of L. It is convenient to write $\vdash_L A$ (or just $\vdash$, if it is clear what L is) to indicate that A is a thesis of L (even though historically the turnstile was originally used to indicate that a formula was a theorem provable in an axiom system).

Proposition. The following conditions are equivalent, for any logic L:

(i) L provides (1), (2), and (RC).
(ii) L provides (1), (2), and (RE).
(iii) L provides (1), (2), and (RPE).
(iv) L provides (K) and (RN).
(v) L provides (RS).

A logic is *normal* if it provides any one of the five conditions of the lemma.

Let us define $L_1 \uplus L_2 = \bigcap\{L : L \supseteq L_1 \ \& \ L \supseteq L_2\}$. In general, define $\biguplus_{i \in I} L_i = \bigcap\{L : \forall i \in I \ L \supseteq L_i\}$.

Proposition. If $\{L_i : i \in I\}$ is a family of [normal] logics, then so are $\bigcap_{i \in I} L_i$ and $\biguplus_{i \in I} L_i$. In particular, if L_1 and L_2 are [normal] logics, then so are $L_1 \cap L_2$ and $L_1 \uplus L_2$.

Let $\mathfrak{S}$ be an axiom system which contains as axioms "sufficiently many" tautologies and has (MP) among its rules and which also satisfies one of the following conditions: (*i*) (1) and (2) are axiom schemata and (RC) is a rule, (ii) (1) and (2) are axiom schemata and (RE) is a rule, (iii) (1) and (2) are axiom schemata and (RPE) is a rule, (iv) (K) is an axiom schema and (RN) is a rule, (v) (RS) is a rule. If Σ is a set of formulæ, then by $\mathfrak{S}\Sigma$ we shall understand the axiom system obtained by adding to the axioms of $\mathfrak{S}$ all formulæ in Σ as new axioms. Notice that if Σ is closed under uniform substitution, then so is the set $\Lambda(\mathfrak{S}\Sigma)$. In other words, the set $\Lambda(\mathfrak{S}\Sigma)$ is a normal logic. Notice that the theorems of $\mathfrak{S}\Sigma$ are the theses of $\Lambda(\mathfrak{S}\Sigma)$, and conversely.

This means that there is a close connexion between axiom systems of this kind and normal logics. Every axiom system of the kind mentioned generates a normal logic. Conversely, given a logic it may be possible to axiomatize it, and in more ways than one.

Again, if Σ is a set of formulæ, then by $\Lambda\Sigma$ we shall understand the smallest normal logic that includes Σ. Notice that if Σ is closed under substitution, then so is $\Lambda\Sigma$.

Proposition. If Σ is closed under uniform substitution, then $\Lambda(\mathfrak{S}\Sigma) = \Lambda\Sigma$.

This means that there are two ways of describing a normal logic that is axiomatizable by an axiom system of the kind we have indicated. One is *to say that it is the smallest logic to provide certain schemata. The other* is to exhibit an axiom system whose set of theorems is precisely that logic. In the examples that now follow we prefer the former way. However, the reader will have no difficulty of providing suitable axiom systems.

Let us now review some modal logics in the literature. First of all there is the smallest normal logic, usually called K (in honour of Kripke). The three most famous modal logics are C. I. Lewis's S4 and S5 and the Gödel/Feys/vonWright logic T. They may be described as the smallest normal logics to provide, respectively,

in the case of T: the schema (T),
in the case of S4: the schemata (T) and (4),
in the case of S5: the schemata (T), (4) and (5).

A shorter way of describing such logics is by providing their *Lemmon code*. If L is a normal logic and $S_0, \ldots, S_n$ are schemata, then $LS_0 \ldots S_n$ is the smallest normal logic including L and providing $S_0, \ldots, S_n$. Thus

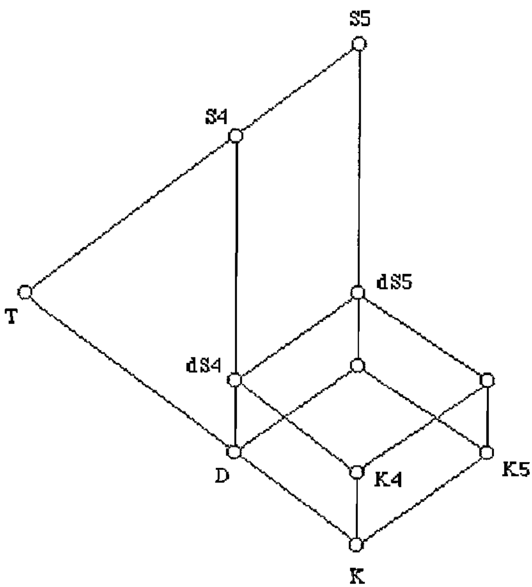
T = KT,
S4 = KT4,
S5 = KT45.

There are of course indefinitely many other ways of describing the same logic. For example, S5 = KT5 = KT4B = KD4B, etc. Other wellknown logics are

the basic deontic logic D = KD,
deontic S4 = KD4,
deontic S5 = KD45,
the Brouwer logic = KTB.

It should be noticed that the labels given here to the various schemata differ in a few places from what one sees in the literature. Thus (D) usually names the schema $\Box A \supset \Diamond A$ and (B) the schema $A \supset \Box\Diamond A$. Lemmon's name for (5) was (E) (for Euclidean).

The relationship of the normal logics definable in terms of the schemata (T), (4), and (5) is set out in the chart below ("Picasso's chair"). Notice that in the chart deontic S4 and deontic S5 are renamed dS4 and dS5, respectively, while KD5 and K45 are not named at all. A logic is included in another if connected to it by a line going up. The eleven logics are all distinct (something we shall be able to prove later).



One reason for preferring to deal with logics rather than axiom systems of Hilbert type—which is what ours are—is that formal proofs in such systems are so cumbersome. Suppose for example that we want to prove that $(\Diamond A \wedge \Box B) \supset \Diamond(A \wedge B)$ is a thesis of K. A natural, informal argument might go like this (where TF stands for truth-functional reasoning and $\vdash$ refers to K):

1.	$\vdash(\neg(A \wedge B) \wedge B) \supset \neg A$	(tautology)
2.	$\vdash(\Box\neg(A \wedge B) \wedge \Box B) \supset \Box\neg A$	(from 1 by Scott's Rule)
3.	$\vdash(\neg\Box\neg A \wedge \Box B) \supset \neg\Box\neg(A \wedge B)$	(from 2 by TF)
4.	$\vdash(\Diamond A \wedge \Box B) \supset \Diamond(A \wedge B)$	(from 3 by df$\Diamond$)

This establishes the desired result. It is important to notice that the informal argument just given is not itself a formal proof, even though a proper formal proof can be found with its help. By the same token, line 4 is not a formal theorem, even though it establishes that $(\Diamond A \wedge \Box B) \supset \Diamond(A \wedge B)$ is a formal theorem of any axiom system whose set of theorems is a normal logic.

4. Maximal consistent sets

A logic L is *finitary* if there exists some axiom system $\mathfrak{S}$, all of whose rules are finitary, such that $L = \Lambda\mathfrak{S}$. With finitary logics the following bit of symbolism is useful. We have already introduced the turnstile $\vdash$ (or $\vdash_L$) to symbolize thesishood in L. We now extend its use by allowing ourselves to write $\Sigma \vdash B$ if there are m and $A_0, ..., A_{m-1} \in \Sigma$ such that $(A_0 \wedge ... \wedge A_{m-1}) \supset B$ is a thesis of L. If m = 0, this means that $\top \supset B \in L$ and hence $B \in L$. Notice that this wider use of the turnstile is in agreement with our previous usage: $\emptyset \vdash_L A$ (in the new sense) if and only if $\vdash_L A$ (in the old sense).

A logic L is *(absolutely) inconsistent* if every formula is a thesis of L, *(absolutely) consistent* otherwise. A set Σ of formulæ is *L-inconsistent* if $\Sigma \vdash \bot$ in L, otherwise *L-consistent*. Thus if L is a finitary logic, Σ is L-inconsistent if and only if for some m there are some formulæ $A_0, ..., A_{m-1} \in \Sigma$ such that $\neg(A_0 \wedge ... \wedge A_{m-1}) \in L$. (Our policy on empty conjunctions and disjunction is that the former collapse to $\top$, the latter to $\bot$. Thus when m = 0 the condition that $\neg(A_0 \wedge ... \wedge A_{m-1}) \in L$ reduces to the condition that $\neg\top \in L$; in other words, that $\bot \in L$.) Notice that if L is inconsistent, then every set of formulæ is L-inconsistent.

Lemma. Let L be any finitary logic. Then a set if L-consistent if and only if each of its finite subsets is L-consistent.

The following is a list of some useful properties of $\vdash$ which hold in any logic (as defined here). Let us agree to write $\Sigma, A_0, \ldots, A_{m-1} \vdash B$ if $\Sigma \cup \{A_0, \ldots, A_{m-1}\} \vdash B$.

$\Sigma \vdash A$ if $A \in \Sigma$ (reflexivity),
$\Sigma \vdash A$ implies $\Sigma' \vdash A$ if $\Sigma \subseteq \Sigma'$ (monotonicity),
$\Sigma \vdash A$ if $\Sigma \vdash C$ and $C \vdash A$ (cut).

$A \wedge B \vdash A$ and $A \wedge B \vdash B$,
$A, B \vdash A \wedge B$;

$\Sigma \vdash A \vee B$ implies $\Sigma \vdash C$ if $\Sigma, A \vdash C$ and $\Sigma, B \vdash C$,
$A \vdash A \vee B$ and $B \vdash A \vee B$;

$A \supset B, A \vdash B$,
$\Sigma, A \vdash B$ implies $\Sigma \vdash A \supset B$;

$A \equiv B, A \vdash B$ and $A \equiv B, B \vdash A$,
$\Sigma, A \vdash B$ and $\Sigma, B \vdash A$ implies $\Sigma \vdash A \equiv B$;

$\neg A, A \vdash B$,
$\Sigma, A \vdash \neg A$ implies $\Sigma \vdash \neg A$;

$\vdash A \vee \neg A$.

Furthermore, in any normal logic we also have

$\Sigma \vdash A$ implies $\Box\Sigma \vdash \Box A$,

where $\Box\Sigma = \{\Box C : C \in \Sigma\}$.

A set Σ of formulæ is *maximal L-consistent* if Σ is L-consistent and, for every L-consistent set Σ', if $\Sigma \subseteq \Sigma'$ then $\Sigma = \Sigma'$. Equivalently, an L- consistent set Σ is maximal L-consistent if, for every $A \notin \Sigma$, the set $\Sigma \cup \{A\}$ is L-inconsistent.

Lemma on Maximal L-Consistent Sets. If L is any logic and Σ is a maximal L-consistent set, then the following conditions obtain:

(i) if $\Sigma \vdash A$ then $A \in \Sigma$,
(ii) $L \subseteq \Sigma$,
(iii) $A \wedge B \in \Sigma$ if and only if $A \in \Sigma$ and $B \in \Sigma$,
(iv) $A \vee B \in \Sigma$ if and only if $A \in \Sigma$ or $B \in \Sigma$,
(v) $A \supset B \in \Sigma$ if and only if, if $A \in \Sigma$ then $B \in \Sigma$,
(vi) $A \equiv B \in \Sigma$ if and only if, $A \in \Sigma$ if and only if $B \in \Sigma$,
(vii) $\neg A \in \Sigma$ if and only if $A \notin \Sigma$.

Proof. Suppose $A \notin \Sigma$. Then, because of maximality, $\Sigma, A \vdash \bot$ in L. If $\Sigma \vdash A$, then $\Sigma \vdash \bot$ by cut, which is impossible if Σ is L-consistent. This proves (i). Clauses (ii)-(vi) are easy consequences of this result. As for (vii), if $\neg A \in \Sigma$ and $A \in \Sigma$ then Σ would obviously be L-inconsistent. On the other hand, if $\neg A \notin \Sigma$ and $A \notin \Sigma$, then by maximality both $\Sigma, \neg A \vdash \bot$ and $\Sigma, A \vdash \bot$. Hence $\Sigma, A \vee \neg A \vdash \bot$. But $\vdash A \vee \neg A$, so by cut $\Sigma \vdash \bot$, contradicting the assumption that Σ is L-consistent.

Do maximal, L-consistent sets exist? Yes, in great abundance, as shown by the following classic result (which proves more than that):

Lindenbaum's Lemma. Let L be any finitary logic, not necessarily normal. Let Σ be any L-consistent set for which it is not the case that $\Sigma \vdash A$. Then there is a maximal L-consistent set Σ^* such that $\Sigma \subseteq \Sigma^*$ and $A \notin \Sigma^*$.

Proof. Let Σ be a given L-consistent set such that $\Sigma \vdash A$ does not hold. Let $A_0, A_1, \ldots, A_n, \ldots$ be an exhaustive enumeration of the formulæ in our language (by the Enumeration Lemma, such an enumeration can be found). Define a family of sets Σ_n as follows:

$$\Sigma_0 = \Sigma,$$

$$\Sigma_{n+1} = \begin{cases} \Sigma_n \cup \{A_n\}, & \text{if this set is L-consistent,} \\ \Sigma_n, & \text{otherwise.} \end{cases}$$

As an inductive argument shows, Σ_n is L-consistent, for every n. Finally define

$$\Sigma^* = \bigcup_{n \geq 0} \Sigma_n.$$

If Σ^* were L-inconsistent then, since L is finitary, some finite subset $\Theta \subseteq \Sigma^*$ would also be L-inconsistent. By the way we have constructed Σ^* there must then be some m such that $\Theta \subseteq \Sigma_m$. But sets including L-inconsistent subsets are themselves L-inconsistent, and Σ_m, we said, is L-consistent. Therefore Σ^* must be L-consistent.

Suppose that Γ is any L-consistent set such that $\Sigma^* \subseteq \Gamma$. Take any formula $B \in \Gamma$. Our enumeration is exhaustive, so there is some m such that $B = A_m$. It follows from our assumption that $\Sigma_m \subseteq \Gamma$, hence we may conclude that $\Sigma_m \cup \{B\} \subseteq \Gamma$. Subsets of L-consistent sets are themselves L-consistent, so $\Sigma_m \cup \{B\}$ is L-consistent. Consequently $B \in \Sigma_m$ and therefore $B \in \Sigma^*$. This shows that $\Gamma \subseteq \Sigma^*$. Hence Σ^* is maximal L-consistent.

The proof of the theorem is now complete, for by construction $\Sigma \subseteq \Sigma^*$.

Corollary. For any consistent finitary logic L there are $2^{\aleph_0}$ maximal L-consistent sets.

Proof. Since any set of formulæ is a subset of Φ, which is known to be denumerable, it is clear that there are at most $2^{\aleph_0}$ sets of formulæ and a fortiori at most $2^{\aleph_0}$ sets that are maximal L-consistent. To see that this upper bound is in fact attained we reason as follows. If A is any set of propositional letters, let us write Π_A for the set $A \cup N$, where $N = \{\neg P :$ P is a propositional letter not in A$\}$. Suppose, for any particular A, that Π_A is L-inconsistent. Then there are some finite subsets $A_0 \subseteq A$ and $N_0 \subseteq N$ such that $A_0 \cup N_0 \vdash \bot$ in L. Now L is closed under uniform substitution. This means that if we substitute $\top$ for all the propositional letters in A_0 and $\bot$ for all those in N_0, then we get $\top \vdash \bot$ in L. In other words, L in absolutely inconsistent, contrary to assumption. Hence Π_A is L-consistent. By Lindenbaum's Lemma it can be extended to a maximal L-consistent set. It is clear that if $A \neq B$, then Π_A and Π_B give rise to incompatible extensions. Since there are $2^{\aleph_0}$ sets of propositional letters, the theorem is proved.

5. Theories

A *theory* is a set of formulæ that contains all tautologies and is closed under (MP). If L is a logic, then an L-*theory* is a logic including L. If L is a normal logic, then an L-theory is *normal* if it includes L and is closed under (RC) (or, equivalently, under (RN) or (RS) or (RE) or (RPE)). Notice that logics are theories closed under uniform substitution. Moreover, in finitary logics L, the maximal L-consistent sets are L-theories, in fact maximal in the same sense as the sets. Thus sets and theories and logics are increasingly specific entities of the same kind. Note that the intersection of any set of [normal] L-theories is a [normal] L-theory.

Let L be a finitary logic and Σ a set of formulæ. Then we define $Cn_L \Sigma = \{A : \Sigma \vdash_L A\}$ (Cn after "consequence", A being seen as a consequence in L if $\Sigma \vdash_L A$). It is clear that $Cn_L \Sigma$ is a theory; in fact, $Cn_L \Sigma$ is a logic (normal if L is normal) if Σ is closed under uniform substitution. Note the following facts:

$$\Sigma \subseteq Cn_L \Sigma,$$
$$Cn_L \Sigma \subseteq Cn_L \Sigma', \text{ if } \Sigma \subseteq \Sigma',$$
$$Cn_L\, Cn_L \Sigma \subseteq Cn_L \Sigma.$$

Proposition. Let L be a finitary logic. Then $Cn_L \Sigma = \bigcap\{T : T \text{ is an L-theory } \& \ \Sigma \subseteq T\}$. In other words, $Cn_L \Sigma$ is the smallest L-theory to include Σ.

Proof. First suppose that $A \in Cn_L \Sigma$. Then $\Sigma \vdash_L A$, and hence there are some formulæ $C_0, ..., C_{n-1} \in \Sigma$ such that $(C_0 \wedge ... \wedge C_{n-1}) \supset A$ is a thesis of L. Let T be any L-theory including Σ. Since $L \subseteq T$ we have $((C_0 \wedge ... \wedge C_{n-1}) \supset A) \in T$. Hence $A \in T$.

Next suppose $A \notin Cn_L \Sigma$. Then it is not the case that $\Sigma \vdash_L A$, so by Lindenbaum's Lemma there is some L-theory including Σ such that $A \notin T$. ◆

We are now able to articulate an important distinction concerning inference rules. If Γ is a set of formulæ and ρ is a rule, let us write Γ^ρ for the closure of $L \cup \Gamma$ under ρ; hence $Cn_L \Gamma = \Gamma^{MP}$. Then we say that ρ is *truth-preserving* in L if, for every set Σ, $\Sigma^\rho \subseteq Cn_L \Sigma$, and that ρ is *validity-preserving* in L if $\varnothing^\rho \subseteq Cn_L \varnothing$. It is clear that (MP) is truth-preserving and that the other rules discussed in section 1.1.2 are validity-preserving. Truth-preserving rules are of course also validity-preserving but, as we shall see in section 1.2.3 below, the converse is not true in general.

1.2. Semantics

1. Model theoretical semantics

A *frame* is a pair (U, R) where U is any set and R is any binary relation on U (that is, $R \subseteq U \times U$). The set U is the *universe*, while R is the *accessibility relation* or *alternativeness relation* of the frame. A *valuation* in U is a function from the set of propositional letters to $\mathfrak{P}U$, the power set of U. A *model* on a frame (U, R) is a triple (U, R, V) where V is a valuation in U.

The central concept in semantics is that of a formula being true at a point in a model, a concept we now define inductively. Let $\mathfrak{M} = (U, R, V)$ be a model. Read "A is true at x in $\mathfrak{M}$" if it is the case that $\mathfrak{M} \models_x A$ and "A is false at x in $\mathfrak{M}$" if it is not the case that $\mathfrak{M} \models_x A$, where A is assumed to be a formula and x an element of U. The basic part of the inductive definition consist of a clause for propositional letters P:

1. $\mathfrak{M} \models_x P$ iff $x \in V(P)$.

The inductive part contains one clause for every Boolean operator that is primitive in our language. We have not specified what they are, but the idea is to articulate the ordinary truth-tables in the present idiom. For example, if conjunction and negation are primitive, then we stipulate

2. $\mathfrak{M} \models_x A \wedge B$ iff $\mathfrak{M} \models_x A$ and $\mathfrak{M} \models_x B$,
3. $\mathfrak{M} \models_x \neg A$ iff it is not the case that $\mathfrak{M} \models_x A$.

In addition the inductive part contains a clause for the only primitive modal operator, necessity:

4. $\mathfrak{M} \models_x \Box A$ iff, for all y, if $(x,y) \in R$ then $\mathfrak{M} \models_y A$.

This ends the definition. If it is not the case that $\mathfrak{M} \models_x A$, we may say that A is *false* at x in $\mathfrak{M}$. Notice that appropriate truth-conditions can be derived for any nonprimitive Boolean operator, for example,

$\mathfrak{M} \models_x A \vee B$ iff $\mathfrak{M} \models_x A$ or $\mathfrak{M} \models_x B$,
$\mathfrak{M} \models_x A \supset B$ iff, if $\mathfrak{M} \models_x A$ then $\mathfrak{M} \models_x B$,
$\mathfrak{M} \models_x A \equiv B$ iff, iff $\mathfrak{M} \models_x A$ then $\mathfrak{M} \models_x B$,
$\mathfrak{M} \models_x \top$,
$\mathfrak{M} \not\models_x \bot$.

Given that we have defined $\Diamond A$ as $\neg\Box\neg A$ we can also derive the truth-condition for that other modal operator, possibility:

$$\mathfrak{M} \models_x \Diamond A \text{ iff, for some y, } (x,y) \in R \text{ and } \mathfrak{M} \models_y A.$$

We say A is *true in* a model $\mathfrak{M}$, in symbols $\mathfrak{M} \models A$, if A is true at every point in the universe of the model; that A is *valid at* a point x in a frame $\mathfrak{F}$, in symbols $\mathfrak{F} \models_x A$, if A is true at x in every model on $\mathfrak{F}$; that A is *valid in* a frame $\mathfrak{F}$, in symbols $\mathfrak{F} \models A$, if it is valid at every point in the universe of the frame. If every thesis of a logic L is true in a model $\mathfrak{M}$ we say that $\mathfrak{M}$ is a *model for* L. If every thesis of L is valid in a frame $\mathfrak{F}$, then we say that $\mathfrak{F}$ is a *frame for* L. A *countermodel* for L of A is a model for L in which A is not true. A *counterframe* for L of A is a frame for L in which A fails to be valid.

We also say that a set Σ is *satisfied* at x in a model $\mathfrak{M}$ if every formula in Σ is true at x in $\mathfrak{M}$; that Σ is *satisfiable* in a frame $\mathfrak{F}$ if Σ is satisfied at some point in some model on $\mathfrak{F}$; and that Σ is *satisfiable* in a class $\mathcal{C}$ of frames if Σ is satisfiable in some frame in $\mathcal{C}$. Note the following facts:

$\Theta(\mathfrak{M}, x) = \{A : \mathfrak{M} \models_x A\}$ is a theory,
$\Theta(\mathfrak{M}) = \{A : \mathfrak{M} \models A\}$ is a normal theory,
$\Lambda(\mathfrak{F}, x) = \{A : \mathfrak{F} \models_x A\}$ is a logic,
$\Lambda(\mathfrak{F}) = \{A : \mathfrak{F} \models A\}$ is a normal logic.

Suppose that $\mathcal{M}$ is any class of models, $\mathcal{F}$ any class of frames. Then we also have

$\Theta(\mathcal{M}) = \bigcap_{\mathfrak{M}\in\mathcal{M}}\Theta(\mathfrak{M})$ is a normal theory,
$\Lambda(\mathcal{C}) = \bigcap_{\mathfrak{F}\in\mathcal{F}}\Lambda(\mathfrak{F})$ is a normal logic.

The only difficulty in establishing these claims is to show that the sets claimed to be logics are closed under uniform substitution. By way of example, let us show that $\Lambda(\mathfrak{F}, x)$ is closed under (US). Suppose that $\mathfrak{F} = (U, R)$ is a frame and that $x \in U$. Assume that $A \in \Lambda(\mathfrak{F}, x)$. Let s be any substitution function. If V is any valuation in U, then we shall write V^s for the valuation assigning V(sP) to each propositional letter P, and $\mathfrak{M}^s$ for the model defined on $\mathfrak{F}$ by V^s. It is straightforward to prove, by induction on A, that

$$\mathfrak{M} \models_x sA \text{ iff } \mathfrak{M}^s \models_x A.$$

From the assumption that A is valid at x in $\mathfrak{F}$ it follows that $\mathfrak{M}^s \models_x A$. Hence, by the result just cited, $\mathfrak{M} \models_x sA$. Therefore $sA \in A(\mathfrak{F}, x)$, as we wanted to show.

In this connexion we might explain our choice of terminology when, in section 1.1.5, we defined some rules as "truth-preserving" or "validity-preserving".

Proposition. (i) Let ρ be any truth-preserving rule. Suppose that every formula in some set Σ is true at an element x in a model $\mathfrak{M}$. Then every formula in Σ^ρ is also true at x in $\mathfrak{M}$. (ii) Let ρ be any validity-preserving rule. Suppose that every formula in a set Σ is valid in a frame $\mathfrak{F}$. Then every formula in Σ^ρ is also valid in $\mathfrak{F}$.

The following result, though simple, is occasionally useful. Let us say that $\mathfrak{M}^t = (U^t, R^t, V^t)$ is the *submodel* of $\mathfrak{M} = (U, R, V)$ *generated* by some element $t \in U$ if

$$U^t = \{x \in U : (t,x) \in R^*\},$$
$$R^t = R \cap (U^t \times U^t),$$
$$V^t(P) = V(P) \cap U^t, \text{ for every propositional letter } P.$$

(Here R^* denotes the ancestral of R, that is, the smallest reflexive, transitive relation to include R.)

Generation Theorem. For all formulæ A and all elements $x \in U^t$,

$$\mathfrak{M}^t \models_x A \text{ if and only if } \mathfrak{M} \models_x A.$$

*2. Algebraic semantics

We say that $\mathfrak{A} = (\mathbf{A}, \cap, \cup, -, \mathbf{0}, \mathbf{1}, \boxdot)$ is a *normal modal algebra* if $(\mathbf{A}, \cap, \cup, -, \mathbf{0}, \mathbf{1})$ is a Boolean algebra and $\boxdot$ is an extra operator from $\mathbf{A}$ to $\mathbf{A}$ such that, for all $\mathbf{a}, \mathbf{b} \in \mathbf{A}$,

$$\boxdot(\mathbf{a} \cap \mathbf{b}) = \boxdot\mathbf{a} \cap \boxdot\mathbf{b},$$
$$\boxdot\mathbf{1} = \mathbf{1}.$$

Let V be an *assignment* in $\mathbf{A}$, that is, a function that assigns to each propositional letter an element of $\mathbf{A}$. We lift V to a function V^* defined on the set of all formulæ as follows. (We have not specified which

Boolean operators are primitive in our language, but assume for the purposes of this definition that conjunction, disjunction and negation are.)

$$V^*(P) = V(P), \text{ for every propositional letter } P,$$
$$V^*(A \wedge B) = V^*(A) \cap V^*(B),$$
$$V^*(A \vee B) = V^*(A) \cup V^*(B),$$
$$V^*(\neg A) = -V^*(A),$$
$$V^*(\Box A) = \boxdot V^*(A).$$

We say that A is *valid* in $\mathfrak{A}$ if, for every assignment in the universe of $\mathfrak{A}$, $V^*(A) = \mathbf{1}$. Let us write $\Lambda(\mathfrak{A})$ for the set of formulæ valid in $\mathfrak{A}$. It is clear that $\Lambda(\mathfrak{A})$ is a normal logic.

Define a dual operator $\Diamond$ by the condition that $\Diamond\mathbf{a} = -\boxdot-\mathbf{a}$, for all $\mathbf{a}, \mathbf{b} \in \mathbf{A}$. Then

$$\Diamond(\mathbf{a} \cup \mathbf{b}) = \Diamond\mathbf{a} \cup \Diamond\mathbf{b},$$
$$\Diamond\mathbf{0} = \mathbf{0}.$$

Notice that, for any valuation V, $V^*(\Diamond A) = \Diamond V^*(A)$.

It is worth noting that for every frame $\mathfrak{F} = (U, R)$ there is an algebra $\mathfrak{A}(\mathfrak{F})$ with the same associated normal logic: define $\mathfrak{A}(\mathfrak{F}) = (\mathfrak{P}U, \cap, \cup, -, \emptyset, U, \mathbf{i})$, where $\mathfrak{P}U$ is the power set of U, and $\cap$, $\cup$ and $-$ are the set theoretical intersection, union and complement with respect to U, respectively, and $\mathbf{i}$ (a kind of "interior operation") is defined by the condition that, for every $X \subseteq U$,

$$\mathbf{i}X = \{x \in U : \forall y\, ((x,y) \in R \Rightarrow y \in X)\}.$$

That $\Lambda\mathfrak{F} = \Lambda\mathfrak{A}(\mathfrak{F})$ is obvious, but it may be instructive to spell out the proof. For each model $\mathfrak{M}$ and formula A, let us call $\|A\|^{\mathfrak{M}} = \{x : \mathfrak{M} \vDash_x A\}$ the *truth set* of A in $\mathfrak{M}$. The following is a way of rewriting the model theoretical truth-conditions listed in section 1.2.1 above:

$$\|P\|^{\mathfrak{M}} = V(P), \text{ for every propositional letter } P,$$
$$\|A \wedge B\|^{\mathfrak{M}} = \|A\|^{\mathfrak{M}} \cap \|B\|^{\mathfrak{M}},$$
$$\|\neg A\|^{\mathfrak{M}} = U - \|A\|^{\mathfrak{M}},$$
$$\|\Box A\|^{\mathfrak{M}} = \mathbf{i}\|A\|^{\mathfrak{M}}.$$

Notice that V is a valuation in U if and only if V is an assignment in $\mathfrak{P}U$. It follows, by an obvious inductive argument, that $\|A\|^{\mathfrak{M}} = V^*(A)$, for all

A. Hence A is valid in $\mathfrak{F}$ iff $\|A\|^{\mathfrak{M}} = U$, for all models $\mathfrak{M}$ on $\mathfrak{F}$ iff $V^*(A) = U$ iff A is valid in $\mathfrak{A}(\mathfrak{F})$.

3. Soundness and completeness

We have seen two different ways of identifying a normal logic: by providing an axiom system and by providing a class of frames (a third way, that of providing a modal algebra, we shall not touch upon). The relationship between these ways is of great interest to modal logicians. Let us say that a normal logic L is *sound* with respect to a class C of frames if $L \subseteq \Lambda(C)$, and *complete* with respect to C if $L \supseteq \Lambda(C)$. If L is both sound and complete with respect to C, and thus $L = \Lambda(C)$, then we say that L is *determined* by C. Sometimes one says that L is *complete* (with no qualification) if L is determined by some class of frames.

It is easy to derive soundness results for all the logics mentioned in section 1.1.3. To begin with, it follows from our discussion in section 1.2.1 that K is sound with respect to the class of all frames. For the others, we can obtain soundness results by imposing conditions on the accessibility relation. We use the following terminology with respect to a frame (U, R), where the quantifiers range over U:

> R is *serial* iff $\forall x\, \exists y\, (x,y) \in R$,
> R is *reflexive* iff $\forall x\, (x,x) \in R$,
> R is *symmetric* iff $\forall x\, \forall y\, ((x,y) \in R \Rightarrow (y,x) \in R)$,
> R is *transitive* iff $\forall x\, \forall y\, \forall z\, (((x,y) \in R\ \&\ (y,z) \in R)) \Rightarrow (x,z) \in R)$,
> R is *Euclidean* iff $\forall x\, \forall y\, \forall z\, (((x,y) \in R\ \&\ (x,z) \in R)) \Rightarrow (y,z) \in R)$.

It is readily shown that KD, KT, KB, KT, K5 are sound with respect to the class of frames that are serial, reflexive, symmetric, transitive, Euclidean, respectively. One consequence of this result is the following:

Proposition. The rules (RC), (RE), (RPE), (RS) and (US) are not truth-preserving in any of the eleven logics in the chart in section 1.1.3.

It is also easy to show that each of the soundness results mentioned is maximal in the sense that no larger class of frames yields soundness. Furthermore, the results are additive in a sense made clear by the following general remark:

Proposition. If L_1 and L_2 are normal logics sound with respect to some classes C_1 and C_2 of frames, then $L_1 \uplus L_2$ is sound with respect to $C_1 \cap C_2$. In general, if $\{L_i\}_{i \in I}$ is a class of logics, for some nonempty index set I, such that each L_i is sound with respect to some class C_i, then $\biguplus_{i \in I} L_i$ is sound with respect to $\bigcap_{i \in I} C_i$.

Corollary. The eleven logics in the chart in section 1.1.3 are all distinct.

Proof. As an example we show that S4 ≠ S5. We already know that S4 is sound with respect to the class of reflexive transitive frames. We also know that $P \vee \Box\neg\Box P$ is a thesis of S5, for every propositional letter. It is easy to find a reflexive transitive frame in which this formula is false at some point. For example, if $U = \{0, 1\}$ and $R = \{(0,0), (0,1), (1,1)\}$, then this formula is false at 0 under any valuation V such that $V(P) = \{1\}$; and this frame is certainly reflexive and transitive. Hence $P \vee \Box\neg\Box P$ is not a thesis of S4. ∎

Notice that if a logic L is sound with respect to a class C of frames, then it is sound with respect to any subclass of C; for it is a general fact that, if A and B are any classes of frames, then $A \subseteq B$ implies that $\Lambda(A) \supseteq \Lambda(B)$. Thus a soundness result is more interesting the stronger the determining class is. Consequently, the most interesting soundness result is one in which the class in question is maximal. This would the case when $C = \{\mathfrak{F} : \forall A \in L\ \mathfrak{F} \models A\} = \{\mathfrak{F} : L \subseteq \Lambda(\mathfrak{F})\}$. In other words, a normal logic L is complete if and only if it is determined by the class of its frames.

The completeness problem we have been discussing so far consists in proving or disproving, of a certain normal logic L and a certain set C of frames, the following claim:

for all A, $A \in L$ if and only if A is valid in C.

An equivalent way of formulating this problem is to ask whether it is true that

for every finite set Σ, Σ is consistent in L if and only if Σ is satisfiable in C.

The kind of completeness we have here is sometimes called weak completeness in order to distinguish it from strong completeness. The latter we define as follows: L is *strongly complete* with respect to $\mathcal{C}$ if every L-consistent set is satisfiable in $\mathcal{C}$. Say that L is *strongly determined* by $\mathcal{C}$ if L is sound and strongly complete with respect to $\mathcal{C}$. Obviously, strong completeness implies weak completeness. The strong completeness problem is to prove or disprove the claim that

$$\text{for every set } \Sigma,\ \Sigma \text{ is consistent in L if and only if } \Sigma \text{ is satisfiable in } \mathcal{C}.$$

4. Canonical models

One of the most powerful techniques for proving completeness is with the help of canonical models. If L is any normal logic, then we define the canonical model for L as the triple $\mathfrak{M}_L = (U_L, R_L, V_L)$, where

U_L = the set of all maximal L-consistent sets,
$R_L = \{(x,y) : x, y \in U_L \ \&\ \forall C\ (\Box C \in x \Rightarrow C \in y)\}$,
$V_L(P) = \{x : x \in U_L \ \&\ P \in x\}$, for every propositional letter P.

In this remarkable model, truth-at-a-point coincides with membership:

Canonical Model Theorem. Let L be any finitary normal logic. For all formulæ A it holds that for all elements $x \in U_L$,

$$\mathfrak{M}_L \models_x A \text{ if and only if } A \in x.$$

Proof. By induction on A. The basic step is a direct consequence of the definition of V_L. For the inductive step, assume that the result holds for A and B. The Boolean parts of this step are easy, but it is instructive to go through one of them, for conjunction, say, to see exactly how it works:

$\mathfrak{M}_L \models_x A \wedge B$ iff (by the truth-condition for $\wedge$)
$\mathfrak{M}_L \models_x A$ and $\mathfrak{M}_L \models_x B$ iff (by the induction hypothesis)
$A \in x$ and $B \in x$ iff (by the Lemma on Maximal L-Consistent Sets)
$A \wedge B \in x$.

Every part of the inductive step repeats these appeals to the appropriate truth-condition, the induction hypothesis and the Lemma on Maximal L-Consistent Sets. This is true also of the step for the necessity, even though that step is more complicated. This is how it begins:

$\mathfrak{M}_L \models_x \Box A$ iff (by the truth-condition for $\Box$)
$\forall y\ ((x,y) \in R_L \Rightarrow \mathfrak{M}_L \models_y A)$ iff (by the induction hypothesis)
$\forall y\ ((x,y) \in R_L \Rightarrow A \in y)$.

Thus the last bit that needs to be proved is that

(†) $\Box A \in x$ if and only if $A \in y$ for all y such that $(x,y) \in R_L$.

First suppose that $\Box A \in x$. If $(x,y) \in R_L$, then it follows from the definition of R_L that $A \in y$. Thus the bit that really needs proving—the only nontrivial part of the entire proof—is the converse. Suppose that $\Box A \notin x$. Consider the set $\Sigma = \{C : \Box C \in x\} \cup \{\neg A\}$. If this set is L-inconsistent then $\Sigma \vdash_L \bot$. Therefore since L is finitary there is some number n and formulæ $C_0, \ldots, C_{n-1}$ such that $C_0 \wedge \ldots \wedge C_{n-1} \in x$ and $(C_0 \wedge \ldots \wedge C_{n-1} \wedge \neg A) \supset \bot$ is a thesis of L. By truth-functional reasoning,

$$\vdash_L (C_0 \wedge \ldots \wedge C_{n-1}) \supset A.$$

Since L is normal, L is closed under Scott's Rule. Hence

$$\vdash_L (\Box C_0 \wedge \ldots \wedge \Box C_{n-1}) \supset \Box A.$$

Applying twice the Lemma on Maximal L-Consistent Sets we conclude, first, that $\Box C_0 \wedge \ldots \wedge \Box C_{n-1} \in x$ and, second, that $\Box A \in x$. This contradicts the assumption that $\Box A \notin x$. The conclusion is that the set Σ is L-consistent. Hence, by Lindenbaum's Lemma, there is some maximal L-consistent set y such that $\Sigma \subseteq y$. Evidently, $A \notin y$ and $y \in U_L$ and $(x,y) \in R_L$ as we wanted. ◆

Notice that the canonical model deserves its name: every thesis of L is true at every point in $\mathfrak{M}_L$, so $\mathfrak{M}_L$ really is a model for L. However, the canonical frame $\mathfrak{F}_L = (U_L, R_L)$ need not be a frame for L. If it is, then let us call L *canonical*. (Here we assume that L is finitary and normal.)

Theorem. Every canonical logic is strongly complete.

Proof. Suppose that Σ is L-consistent. By Lindenbaum's Lemma there is some $w \in U_L$ such that $\Sigma \subseteq w$. By the Canonical Model Theorem, if $A \in w$ then A is true at w in $\mathfrak{M}_L$. Hence Σ is satisfiable in $\mathfrak{F}_L$. In other words, L is strongly determined by its canonical frame. ∎

Thus for every canonical logic we have two completeness results: it is determined by the canonical frame, but of course also by the class of all frames for the logic. In general, there may be any number of classes of frames determining a logic.

The preceding work makes it easy to derive completeness results for all the eleven logics discussed above. For example, to show that the Gödel/Feys/von Wright logic T is determined by the class of reflexive frames, it is enough to show (i) every reflexive frame is a frame for T, (ii) the canonical frame for T is reflexive. The former we already know. To prove the latter, take any $x \in U_T$. By definition of R_T, $(x,x) \in R_T$ if $\forall C\, (\Box C \in x \Rightarrow C \in x))$. Thanks to the axiom schema (T), which is provided by the logic T, this is certainly the case.

5. The finite model property

This section contains an account of some technical concepts, the importance of which will become clear from the following two sections. A model is *separable* if for any two points in the model there is some formula that is true at one of the points but false at the other. Note that canonical models are always separable.

Lemma. Let $\mathfrak{M} = (U, R, V)$ be any model. Then there is a separable model $\mathfrak{M}^\wedge = (U^\wedge, R^\wedge, V^\wedge)$ and a surjection $f : U \longrightarrow U^\wedge$ such that, for all A, $\mathfrak{M} \models_x A$ if and only if $\mathfrak{M}^\wedge \models_{fx} A$.

Proof. Define a relation $\cong$ by the condition $x \cong y$ if and only if, for all A, $\mathfrak{M} \models_x A$ if and only if $\mathfrak{M} \models_y A$. This relation is an equivalence relation. Let $x^\wedge$ be the equivalence class $\{u \in U : x \cong u\}$. Write $U/\cong$ for the set of equivalence classes. Define $\mathfrak{M}^\wedge = (U^\wedge, R^\wedge, V^\wedge)$, where

$U^\wedge = U/\cong$,
$R^\wedge = \{(x^\wedge, y^\wedge) : \exists u \in x^\wedge\, \exists v \in y^\wedge\, (u,v) \in R\}$,
$V^\wedge(P) = \{x^\wedge : \exists u \in x^\wedge\, u \in V(P)\}$, for every propositional letter P.

Define $fx = x^\wedge$, for every $x \in U$. The assertion of the theorem is then proved by induction on A. ■

We say that a set $V \subseteq U$ is *definable by a formula* in a model $\mathfrak{M} = (U, R, V)$ if there is some formula C such that $V = \{x \in U : \mathfrak{M} \models_x C\}$. In this case we say that C is a *characteristic* formula for V.

Lemma. In a finite separable model every subset is definable by a formula.

Proof. Let $\mathfrak{M} = (U, R, V)$ be separable and finite. For each $x, y \in U$ such that $x \neq y$ let $C_{x,y}$ be a formula true at x and false at y (this is possible since $\mathfrak{M}$ is separable). For each $x \in U$ define C_x as the conjunction (in some order) of the formulæ in $\{C_{x,y} : x \neq y\}$. Finally, if $V \subseteq U$, then define C_V as the disjunction (in some order) of the formulæ in $\{C_x : x \in V\}$. (The conjunctions and disjunctions are well-defined since $\mathfrak{M}$ is finite.) It is clear that C_V is true at every point in V and at no other point. ■

A normal logic is said to have the *finite model property (fmp)* if every nonthesis of the logic has a finite countermodel that is a model for the logic. It has the fmp *in the strong sense* if, for every nonthesis A, there is a number n(A) such that A is false in some model for L with at most n(A) elements. Similarly, it has the *finite frame property (ffp)* is every nonthesis has a finite counterframe that is a frame for the logic, and it has the ffp in the strong sense if, for every nonthesis A, there is a number n(A) such that A is false in some model on some frame for L with at most n(A) elements.

Theorem. A normal logic has the fmp if and only if it has the ffp.

Proof. It is at once clear that having the ffp implies having the fmp. For the converse, suppose that L is a normal logic for which $\mathfrak{M} = (U, R, V)$ is a finite model. There is no loss of generality if we assume that $\mathfrak{M}$ is separable. We wish to show that (U, R) is a frame for L. Suppose that V^* is any valuation in U and write $\mathfrak{M}^* = (U, R, V^*)$. It will be enough to show that $\mathfrak{M}^*$ is a model for L. This we do by showing how we can simulate $\mathfrak{M}^*$ within $\mathfrak{M}$. Since $\mathfrak{M}$ is finite and separable we can find, for each propositional letter P, a formula C_P that is characteristic of

$\|P\|^{\mathfrak{M}^*}$, the truth-set of P in $\mathfrak{M}^*$. Let s be the substitution function that assigns, to each propositional letter P, the formula C_P. Then it is easy to prove by induction that for every formula A and every point $x \in U$,

$$\mathfrak{M} \models_x sA \text{ if and only if } \mathfrak{M}^* \models_x A.$$

Suppose now that $A \in L$. Let x be any element of U. Then $sA \in L$ since L is closed under uniform substitution. The assumption that $\mathfrak{M}$ is a model for L implies that sA is true at x in $\mathfrak{M}$. Hence by the result just proved, A is true at x in $\mathfrak{M}^*$.

6. Filtrations

We say that a set Ψ of formulæ is *closed under subformulæ* if, for all n-ary operators o and all formulæ $A_0, \ldots, A_{n-1}$, if $o(A_0, \ldots, A_{n-1}) \in \Psi$ then $A_0, \ldots, A_{n-1} \in \Psi$. Furthermore, A is a *subformula* of B if $A \in \Psi$ where Ψ is the smallest set that contains B and is closed under subformulæ. Note that the set of subformulæ of any given formula is finite.

Let $\mathfrak{M} = (U, R, V)$ be a given model. Then any set Ψ of formulæ closed under subformulæ induces a relation $\equiv (\text{mod } \Psi)$ on U by the condition $x \equiv y \ (\text{mod } \Psi)$ if and only if, for all $A \in \Psi$, $\mathfrak{M} \models_x A$ iff $\mathfrak{M} \models_y A$. It is readily seen that this relation is an equivalence relation. Let us write x° for the equivalence class of x, that is, the set $\{u \in U : x \equiv u \ (\text{mod } \Psi)\}$. (We may drop the reference to Ψ when it is clear what set Ψ is.) Let us write U/Ψ for the class $\{x^\circ : x \in U\}$ of equivalence classes.

Lemma. If Ψ is finite, then U/Ψ is finite.

Proof. If Ψ contains exactly n elements, then U/Ψ cannot contain more than 2^n elements.

By a *filtration* of $\mathfrak{M}$ through Ψ we mean a model $\mathfrak{M}^\circ = (U^\circ, R^\circ, V^\circ)$ satisfying the following conditions:

(i) $U^\circ = U/\Psi$,
(iiA) if $(x,y) \in R$ then $(x^\circ, y^\circ) \in R^\circ$,
(iiB) if $(x^\circ, y^\circ) \in R^\circ$ then, for all A such that $\Box A \in \Psi$, if $\mathfrak{M} \models_x \Box A$ then $\mathfrak{M} \models_y A$,
(iii) if P is a propositional letter such that $P \in \Psi$, then $x^\circ \in V^\circ(P)$ if and only if $x \in V(P)$.

Filtration Theorem. Let $\mathfrak{M}^\circ$ be a filtration of $\mathfrak{M}$ through Ψ. Then, for all formulæ $A \in \Psi$ and all elements x in $\mathfrak{M}$,

$$\mathfrak{M}^\circ \models_{x^\circ} A \text{ if and only if } \mathfrak{M} \models_x A.$$

Proof. By induction on A. Condition (iii) is tailored to suit the basic step. In the inductive step the Boolean cases are trivial, while conditions (iiA) and (iiB) guarantee that the modal case goes through. ■

Note that filtrations always exist. In particular, the following definitions yield filtrations:

$$(x^\circ,y^\circ) \in R_{min} \text{ iff } \exists x' \equiv x\ \exists y' \equiv y\ (x',y') \in R,$$

$$(x^\circ,y^\circ) \in R_{max} \text{ iff } \forall A \in \Psi\ (\mathfrak{M} \models_x \Box A \Rightarrow \mathfrak{M} \models_y A).$$

The former we call a *minimal*, the latter a *maximal* filtration (over U/Ψ). This terminology is not gratuitous, for if R° is any filtration relation over U/Ψ, then $R_{min} \subseteq R^\circ \subseteq R_{max}$. The following result shows that in one important case all filtration relations coincide:

Theorem. Suppose $\mathfrak{M}_L$ is the canonical model for some normal logic L. Let $\mathfrak{M}^\circ = (U/\Psi, R^\circ, V^\circ)$ be a filtration of $\mathfrak{M}_L$ through Ψ. Then, if $\mathfrak{M}^\circ$ is a model for L, then $R_{min} = R^\circ = R_{max}$.

Proof. In view of previous remarks it will be enough to show that $R_{max} \subseteq R_{min}$. Suppose that $(x^\circ,y^\circ) \in R_{max}$. Consider the sets $\Gamma = \{A : \mathfrak{M}^\circ \models_{x^\circ} A\}$ and $\Delta = \{A : \mathfrak{M}^\circ \models_{y^\circ} A\}$. Note the obvious fact that $\Gamma \equiv x$ and $\Delta \equiv y \pmod{\Psi}$. Since, by assumption, $\mathfrak{M}^\circ$ is a model for L, Γ and Δ are L-consistent, indeed maximal L-consistent, so $\Gamma, \Delta \in U_L$. It is easy to verify that $(\Gamma,\Delta) \in R_L$. Hence $(x^\circ,y^\circ) \in R_{min}$. ■

In the important applications of the filtration theorem it is the canonical model (or a generated submodel thereof) that gets filtered through the set of subformulæ of some formula. The minimal normal logic K provides an example. Suppose that A is any nonthesis of K. Then we know that A is false at some element x in the canonical model for K. Let Ψ be the set of subformulæ of A. Then A is false at x° in any filtration of the canonical model through Ψ. Moreover, as Ψ is finite in this case, the filtration is finite and in fact of cardinality at most 2^n, where n is the cardinality of Ψ. Finally, the filtration is surely a model for K (every model is a model for K!). Hence K has the finite model property in the strong sense.

The argument just given can be adapted to show that not only K but in fact all eleven of the logics discussed in section 1.1.3 has the fmp in the strong sense. We end the section by presenting another example, that of S4.

Let $\mathfrak{M}_{S4}$ be the canonical model for S4. We want to show that S4 has the strong fmp. Lemmon and Scott, who were the first to do this, did it by defining $\mathfrak{M}^{\#} = (U_{S4}/\Psi, R^{\#}, V^{\circ})$, where Ψ is any fixed, finite set of formulæ, U_{S4}/Ψ and V° are as above, and $R^{\#}$ is defined by the condition

$$R^{\#} = \{(x^{\circ},y^{\circ}) : \forall A\ (\Box A \in x \cap \Psi \ \Rightarrow\ A \in y\ \&\ \Box A \in y)\}.$$

Then it is immediately clear that $\mathfrak{M}^{\#}$ is a filtration, and since it is finite, reflexive and transitive the desired result has been achieved.

However, for pedagogical reasons we shall now consider a longer proof of the same result, which shows in a simple form a line of reasoning we shall encounter twice below, first in connexion with ancestral and then with dynamic logic. Define $\mathfrak{M}\dagger = (U_{S4}/\Psi, R\dagger, V^{\circ})$ where U_{S4}/Ψ and V° are as before but $R\dagger$ is the ancestral of R_{min}:

$$R\dagger = (R_{min})^*.$$

Thus defined, $\mathfrak{M}\dagger$ is a finite, reflexive and transitive model and therefore a model for S4. Hence if we can show that $\mathfrak{M}\dagger$ is a filtration, then we have shown that S4 possesses the strong fmp. What needs to be done is to show that conditions (iiA) and (iiB) in the definition of filtration are satisfied. This we do in two steps, Lemma A and Lemma B.

Lemma A. If $(x,y) \in R_{S4}$, then $(x^{\circ},y^{\circ}) \in R\dagger$.

Proof. Immediate.

When we go to the next step we make use of the fact that in the canonical model truth-at-a-point and membership are co-extensive properties.

Lemma B. If $(x^\circ, y^\circ) \in R\dagger$ and $\Box A \in x \cap \Psi$, then $A \in y$.

Proof. We begin by proving the following auxiliary contention:

(‡) if $(x^\circ, y^\circ) \in (R_{min})^n$, then $\Box A \in x$ implies $\Box A \in y$.

The proof is by induction on n. If n = 0 the contention is clearly true. Assume that it is true for n (the inductive hypothesis) and also that $(x^\circ, y^\circ) \in (R_{min})^{n+1}$. Then there is some w such that $(x^\circ, w^\circ) \in (R_{min})^n$ and $(w^\circ, y^\circ) \in R_{min}$. Suppose that $\Box A \in x$. By the inductive hypothesis, $\Box A \in w$. Since $(w^\circ, y^\circ) \in R_{min}$ there will be elements w' and y' such that $w \equiv w'$ and $y \equiv y'$ (mod Ψ) and $(w', y') \in R_{S4}$. By assumption $\Box A \in \Psi$, so $\Box A \in w'$. The fact that $w' \in U_{S4}$ and that $\Box A \supset \Box\Box A$ is a thesis of S4. Hence $\Box\Box A \in w'$. Therefore $\Box A \in y'$. Appealing again to the fact that $\Box A \in \Psi$, we conclude that $\Box A \in y$. This ends the proof of the contention (‡).

After this preliminary, assume, for any $x, y \in U_{S4}$, that $(x^\circ, y^\circ) \in R\dagger$ and that $\Box A \in x \cap \Psi$. Then $\Box A \in y$ by (‡). Furthermore, $\Box A \supset A$ is a thesis of S4. Hence $A \in y$, as we wanted. ■

By a theorem proved earlier we see that our relation R† is in fact identical with Lemmon and Scott's relation $R^{\#}$. Their proof is of course much shorter than ours, but it is ad hoc in a way that ours is not. For we know that if we are to succeed in our enterprise—to prove that there exists a filtration of the canonical model for S4 through the given finite set Ψ that is a model for S4—then $\mathfrak{M}\dagger$ must be that filtration. The reason is that R† has to be reflexive and transitive, and it has to include the minimal filtration relation R_{min}. Consequently, R† has to include $(R_{min})^*$. Since there seems to be no further condition to add, R† defined as $(R_{min})^*$ is the natural candidate for a filtration relation.

This is in fact the recipe we shall follow when we are faced with similar problems in chapters 2 and 3: to start with the minimal filtration and then add whatever conditions the situation requires. The finite model we define will therefore always be a model for the relevant logic, and the only difficulty will be to prove that it is a filtration.

7. Decidability

Let us say that a concept is *(effectively) decidable* in a certain domain if for every entity in that domain one can decide, in a finite number of steps, whether the concept applies to that entity. Thus a logic is decidable if for every formula in the language it is possible to decide, in a finite number of steps, whether the formula is a thesis. The concept of being a finite frame for a logic is decidable if, for every finite frame one can decide, in a finite number of steps, whether that frame is a frame for the logic. In modal logic the main usefulness of the filtration method has been in proving the decidability of logics.

Theorem. Let L be a normal logic for which the notion 'finite frame for L' is decidable. Then L is decidable if L has the fmp in the strong sense.

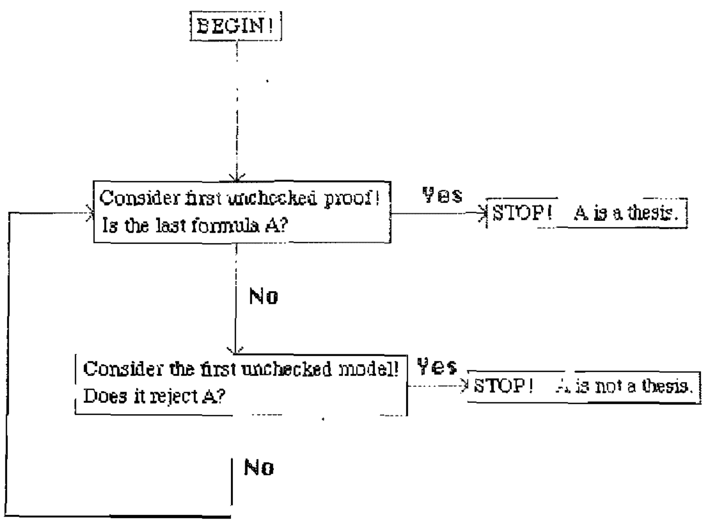
It follows that the eleven logics mentioned in section 1.1.3 are all decidable.

Theorem. Suppose that L is a normal logic that can be axiomatized with only finitely many axiom schemata and with (RC) or (RN) or (RS) or (RPE) as only inference rule other than (MP). Then L is decidable if L has the fmp.

Proof. On the given hypothesis you can devise a method for deciding L by defining the following two projects. One is to provide an enumeration of all formal proofs in the axiom system that generates L, call it $\mathfrak{S}$. Since there are only finitely many axiom schemata in $\mathfrak{S}$ and only the two rules, this is a possible task. The other is to provide an enumeration of all finite frames for L. This can be done by going through all finite models (identifying isomorphic models), first those with one element, then those with two elements, then those with three elements, and so on. For each frame one has to check whether the axiom schemata of $\mathfrak{S}$ are true; this is possible since the model is finite and only has finitely many propositions. The inference rules mentioned need not be checked since we already know they are validated.

With both projects defined, we can proceed to decide whether a given formula A is a thesis of L or not: "all we have to do" (in fact a Gargantuan task) is simultaneously to go through the two enumerations. In the enumeration of formal proofs we look at the last formula of the formal proof: is that the formula A? If the answer is Yes, then we know that A is a thesis. If the answer is No, then we turn to the enumeration of frames for L. This time we ask whether A is false at any point in any model on the frame. As with checking the axiom schemata, this task can be accomplished in a finite number of steps. If the answer is Yes, we know that A is not a thesis. If the answer is No, we go back to the enumeration of formal proofs and begin a new round of investigation. (The flow-chart below should make the procedure quite clear.)

The way we have set up the two projects guarantees that, sooner or later, our quest will terminate. For if A is a thesis, then the assumption that L is axiomatized by $\mathfrak{S}$ implies that there is a formal proof of A, and that proof must then occur in our enumeration of formal proofs. On the other hand, if A is not a thesis, then the assumption that L has the fmp implies that A has a finite counterframe for L, and so that frame occurs in our enumeration of frames. ■



2. Ancestral logic

2.1. Semantics

1. Model theory

In ancestral logic is seems natural to begin with semantics, for that is where the generalization over modal logic takes place. Having defined a more inclusive kind of model we may then ask ourselves what the object language would be appropriate to deal with this new kind of model. Having settled on an object language we may then turn to the question of how to axiomatize various logics defined by the semantics.

In ordinary modal logic frames are pairs (U, R) where U is a set and R is a binary relation in U. One other binary relation definable in terms of R is R*, the ancestral of R. Why not introduce a new modal operator, perhaps written [∗], with the following truth-condition relative to a model $\mathfrak{M}$ and a point x in the universe of the model:

$$\mathfrak{M} \models_x [*]A \text{ iff for all y, if } (x,y) \in R^* \text{ then } \mathfrak{M} \models_y A.$$

This is what is done in ancestral logic, except that we will put our definitions in a slightly different way. By a *frame* (for ancestral logic) we shall mean a triple (U, R, S) where U is a set, and R and S are binary relations in U. As in modal logic, a model is a frame with a supplementary valuation; hence a *model* (for ancestral logic) is a structure (U, R, S, V) where (U, R, S) is a frame and V is a valuation in U.

We now officially add [∗] as a new modal operator to the language of chapter 1. For the sake of typographical consistency we write [·] instead of □. To the truth-conditions for propositional letters and Boolean operators listed in chapter 1 we add the following two, where we assume that $\mathfrak{M}$ = (U, R, S, V) and x ∈ U:

$$\mathfrak{M} \models_x [\cdot]A \text{ iff for all y, if } (x,y) \in R \text{ then } \mathfrak{M} \models_y A.$$

$$\mathfrak{M} \models_x [*]A \text{ iff for all y, if } (x,y) \in S \text{ then } \mathfrak{M} \models_y A.$$

If we introduce $<*>$ as another modal operator by the definition $<*>A = \neg[*]\neg A$, for all A, we can derive the following truth-condition:

$$\mathfrak{M} \models_x <*>A \text{ iff for some } y, (x,y) \in S \text{ and } \mathfrak{M} \models_y A.$$

Concepts of truth and validity, etc., can now be taken over from modal logic. However, this will not give the desired result: our notion of frame is too general. To overcome this problem let us define a frame (U, R, S) or model (U, R, S, V) as *standard* if $S = R^*$. It is clear that every frame, standard or not, determines a logic. The logic of paramount interest to us is that determined by the class of all standard frames; let us call it the *basic ancestral logic*. To axiomatize that logic and prove the axiomatization sound and complete is the task of this chapter.

For any one-place propositional operator o, define o^nA as the formula consisting of A preceded by n occurrences of o. Formally, define $o^0A = A$ and $o^{n+1}A = oo^nA$, for all n. Define R^n as the relative product of R by itself n times. Formally, let Δ_U be the diagonal relation $\{(x,x) : x \in U\}$. Then, if R is a binary relation in U, define $R^0 = \Delta_U$ and $R^{n+1} = R \mid R^n$. (Hence $R^* = \bigcup_{n\geq 0} R^n$.) Note the following truth-condition for $[\cdot]^n$:

$$\mathfrak{M} \models_x [\cdot]^nA \text{ iff for all } y, \text{ if } (x,y) \in R^n \text{ then } \mathfrak{M} \models_y A.$$

Lemma. A frame $\mathfrak{F}$ is standard if and only if all instances of the following schemata are valid in $\mathfrak{F}$:

(∗E)	$[*]A \supset [\cdot]^nA$, for all n,
(∗I)	$(A \wedge [*](A \supset [\cdot]A)) \supset [*]A$.

Proof. It is easy to show that they are valid if $\mathfrak{F}$ is standard. For the converse, suppose that $\mathfrak{F} = (U, R, S)$ provides (∗E), for all n, and also (∗I).

First suppose that there is some pair $(x,y) \in R^* - S$. Let V be a valuation in U assigning $\{u : (x,u) \in S\}$ to a certain propositional letter P. Then the situation is that $(x,y) \notin S$ but there is some n such that $(x,y) \in R^n$. Consequently in the model defined by this valuation $[*]A$ is true at x while $[\cdot]^nA$ is false, contradicting the validity of (∗E). Hence $R^* \subseteq S$.

Next suppose that there is some pair $(x,y) \in S - R^*$. Let V be a valuation assigning $\{u : (x,u) \in R^*\}$ to a certain propositional letter P. Then, since R^* is reflexive,

(1) P is true at x.

Moreover, suppose that $(x,v) \in S$. Say that P is true at v. Then $(x,v) \in R^*$. Moreover, for any w such that $(v,w) \in R$ we have $(x,w) \in R^*$, and so P is true at w. This means that $[\cdot]A$ is true at v, and so

(2) $[*](A \supset [\cdot]A)$ is true at x.

But $(x,y) \in S$ and, since $(x,y) \notin R^*$, P is false at y. Therefore

(3) $[*]A$ is false at x.

This is in contradiction with the assumption that $(*I)$ is valid on $\mathfrak{F}$. Hence $S \subseteq R^*$. ■

2. Compactness and noncompactness

A logic L is *compact* over a class $\mathcal{C}$ of frames if any set Σ of formulæ is satisfiable in $\mathcal{C}$ whenever every finite subset of Σ is satisfiable in $\mathcal{C}$. Let us say that L is compact (without qualification) if L is compact over the class of its frames. This is a general concept and certainly meaningful for modal logics.

Theorem. Let L be a finitary normal modal logic. If L is strongly complete, then L is compact.

Proof. Let L be finitary and strongly determined by $\mathcal{C}$, where $\mathcal{C}$ is the class of its frames. Suppose that Σ is a set of formulæ, every finite subset of which is satisfiable in $\mathcal{C}$. Then every finite subset of Σ is satisfied at some point in some model on some frame in $\mathcal{C}$, and therefore is L-consistent. Hence by finitariness, Σ is L-consistent. Hence by strong completeness, Σ is satisfiable in $\mathcal{C}$. ■

Corollary. Every finitary normal canonical logic is compact.

By contrast we have the following result:

Theorem. The basic ancestral logic is not compact.

Proof. Consider the set $\Sigma = \{[\cdot]^n P : n \geq 0\} \cup \{\neg[*]P\}$, where P is some propositional letter. Let $\mathfrak{F} = (U, R, S)$ be the frame where U is the set of natural numbers, R is the successor relation $\{(n, n+1) : n \geq 0\}$ and $S = R^*$. Then $\mathfrak{F}$ is a standard frame. For any n let V_n be any valuation assigning $\{i : i \leq n\}$ to P.

Let Σ' be any finite subset of Σ. Then there is some m such that, for all $i > m$, $[\cdot]^i P \notin \Sigma'$. Consider the model defined on $\mathfrak{F}$ by V_m. In that model $[\cdot]^i P$ is true at 0 for all $i \leq m$. Moreover, $\neg[*]P$ is true at 0. Hence Σ' is satisfied at 0.

This argument shows that every finite subset of Σ is satifiable in a standard frame. However, it is easy to see that Σ itself is not satisfiable in any standard frame. ∎

2.2. Syntax

1. Axiomatics

Let $\mathfrak{A}$ be the following axiom system: the axioms of $\mathfrak{A}$ are

(i) all instances of any thesis of the modal logic K for $[\cdot]$ and $[*]$,
(ii) all instances of the schemata

$(*T)$	$[*]A \supset A$,
$(*E_1)$	$[*]A \supset [\cdot]A$,
$(*4)$	$[*]A \supset [*][*]A$.
$(*ind)$	$(A \wedge [*](A \supset [\cdot]A)) \supset [*]A$,

and the inference rules of $\mathfrak{A}$ are (MP) and (RN) for $[*]$. (Explanation of the labels: $(*E_1)$ is a kind of elimination schema, $(*ind)$ a kind of "induction" schema. The other two—$(*T)$ and $(*4)$—are straightforward generalizations of the modal schemata (T) and (4).)

We know from considerations in section 1.1.2 that instead of (RN) we might equally well have adopted as a primitive rule (RC), (RS), (RE) for $[*]$ or (RPE). There are two other popular modifications of $\mathfrak{A}$ that likewise do not affect the set of theorems.

Proposition. Let $\mathfrak{A}'$ be the axiom system obtained from $\mathfrak{A}$ by replacing the schemata $(*T)$, $(*E_1)$ and $(*4)$ by the single schema

$$(*M) \qquad [*]A \supset (A \wedge [\cdot][*]A).$$

Then $\Lambda\mathfrak{A}' = \Lambda\mathfrak{A}$.

Proof. That $(*M)$ implies $(*T)$ and $(*E_1)$ is clear. To see that it also *implies* $(*4)$, notice that $(*M)$ implies that $\vdash [*]A \supset [\cdot][*]A$. *Applying (RN), we conclude that* $\vdash [*]([*]A \supset [\cdot][*]A)$. *But the following is an instance of* $(*\mathrm{ind})$:

$$\vdash ([*]A \wedge [*]([*]A \supset [\cdot][*]A)) \supset [*][*]A.$$

Hence by truth-functional reasoning $\vdash [*]A \supset [*][*]A$. ∎

Proposition. Let $\mathfrak{A}''$ be the axiom system obtain from $\mathfrak{A}$ by deleting the schema $(*\mathrm{ind})$ and instead adding the inference rule

$$(\mathrm{R}*\mathrm{ind}) \qquad \{A \supset [\cdot]A\} \longmapsto \{A \supset [*]A\}.$$

Then $\Lambda\mathfrak{A}'' = \Lambda\mathfrak{A}$.

Proof. From proposition just proved, we know that, for every C, $\vdash [*]C \supset [\cdot][*]C$. In particular, then,

$$(1) \qquad \vdash [*](A \supset [\cdot]A) \supset [\cdot][*](A \supset [\cdot]A).$$

Using $(*T)$ and truth-functional reasoning we observe that

$$(2) \qquad \vdash (A \wedge [*](A \supset [\cdot]A)) \supset [\cdot]A.$$

Putting (1) and (2) together we conclude, with a bit of modal reasoning, that

$$\vdash (A \wedge [*](A \supset [\cdot]A)) \supset [\cdot](A \wedge [*](A \supset [\cdot]A)).$$

Hence by the new rule (R$*$ind),

$$\vdash (A \wedge [*](A \supset [\cdot]A)) \supset [*](A \wedge [*](A \supset [\cdot]A)).$$

Consequently, $\vdash (A \wedge [*](A \supset [\cdot]A)) \supset [*]A$.

The choice between these many different possibilities will depend on the purpose of the analysis and perhaps personal taste. Here there is no need to choose. (Question: How does the schema $[*]A \supset (A \wedge [\cdot][*]A)$ compare to ($*$M)?)

Soundness Theorem. The system $\mathfrak{A}$ is sound with respect to the class of standard frames.

By an *ancestral* logic we mean a logic containing as theses all theorems of $\mathfrak{A}$. An ancestral logic is *normal* if it is closed under (RC), (RN), (RS), (RE) for $[*]$ or under (RPE) (if it closed under one it is closed under all). Thus in a normal ancestral logic, $[\cdot]$ is at least a K-modality and $[*]$ at least an S4-modality.

2. Comparison with tense-logic

Basic tense-logic with both future and past time also employs frames (U, R, S), but there the requirement of standardness is that R and S be the inverses of one another; that is, that $S = \{(x,y) : (y,x) \in R\}$, whence of course $R = \{(x,y) : (y,x) \in S\}$. Comparison with this kind of tense-logic is not particularly enlightening, at least not at this point.

More relevant is tense-logic for discrete future time. One naturally thinks of time as linear, but in their abstract analyses tense-logicians like to allow future time to "branch". Thus their frames are triples (U, R, S) where $S = R^*$, as in our standard ancestral frames. In their language they have two non-Boolean operators, often written $\bigcirc$ and $\Box$. An informal reading of $\bigcirc A$ is "at the next moment, A" or "tomorrow, A", while $\Box A$ is read "at every future moment, A" or "always, A".

The characteristic axiom schemata for tense-logics relating to this conception are similar to those of $\mathfrak{A}$. For example, we have K for $\bigcirc$, and parallelling (∗T), (∗E_1), (∗4), and (∗ind) we have either (if time is reflexive so that the future is regarded as including the present)

$$\Box A \supset A,$$
$$\Box A \supset \bigcirc A$$
$$\Box A \supset \Box\Box A,$$
$$(A \wedge \Box(A \supset \bigcirc A)) \supset \Box A,$$

or else (if time is irreflexive so that the future is regarded as beginning tomorrow)

$$\Box A \supset \bigcirc A,$$
$$\Box A \supset \Box\Box A,$$
$$(\bigcirc A \wedge \Box(A \supset \bigcirc A)) \supset \Box A.$$

Usually tense-logicians cast time as being endless. When they do, the following schema becomes valid:

$$\neg(\bigcirc A \wedge \bigcirc\neg A).$$

The case when time is linear is another special case. The following extra axiom schema then becomes appropriate:

$$\bigcirc A \vee \bigcirc\neg A.$$

That addition makes $\bigcirc$ a particularly pleasant operator to work with as it now commutes with every other operator: $\bigcirc\neg A \equiv \neg\bigcirc A$, $\bigcirc(A \wedge B) \equiv (\bigcirc A \wedge \bigcirc B)$, $\Box\bigcirc A \equiv \bigcirc\Box A$, etc., all become theorems.

2.3. Completeness

1. Canonical models

Let L be a finitary normal logic. We can define the canonical model for L along the lines of section 1.2.4, namely, by putting $\mathfrak{M}_L = (U_L, R_L, S_L, V_L)$ where

U_L = the set of all maximal L-consistent sets,
$R_L = \{(x,y) : \forall C\ ([\cdot]C \in x \Rightarrow C \in y)\}$,
$S_L = \{(x,y) : \forall C\ ([*]C \in x \Rightarrow C \in y)\}$,
$V_L(P) = \{x : P \in x\}$, for all propositional letters P.

It is now possible to prove a Canonical Model Theorem to the effect that $\mathfrak{M}_L \models_x A$ if and only if $A \in x$, for all formulæ A and all points $x \in U_L$. However, in contrast with the cases we examined in chapter 1, this canonical model is not immediately useful to us, for although it is a model for the logic, it is not a standard model. It is a fact that $(R_L)^* \subseteq S_L$, but it is also a fact—except when L is one of certain trivial logics—that $(R_L)^* \neq S_L$. For let Σ be the set $\{[\cdot]^n P : n \geq 0\} \cup \{\neg[*]P\}$. In section 2.1.2 we saw that Σ is consistent in any ancestral logic for which the natural numbers frame $\mathfrak{F}$ defined there is a frame. In such logics, by Lindenbaum's Lemma, there exists a maximal L-complete set u extending Σ. Note that $u \in U_L$. Since $[*]P \notin u$ there exists some w such that $(u,w) \in S_L$ and $P \notin w$. Sincce $[\cdot]^n P \in u$, for every n, $(u,w) \notin (R_L)^*$.

What this means is that the canonical frame (U_L, R_L, S_L) is not a frame for L in any interesting case. In other words, no interesting normal ancestral logic is canonical. This in turn means that extra work is needed before completeness can be established. Our strategy will be to proceed via filtrations: filtering the canonical model through a judiciously chosen formula set will produce a filtration that is a standard model suited to our purpose.

2. Filtrations

Filtrations are defined in the same way as in chapter 1. Since L is normal and finitary, the canonical model for L exists with the usual properties. Here we are only going to filter the canonical model of L, so we phrase the definition of filtration with an eye to that particular application. Let Ψ be a set of formulæ that is closed under subformulæ. The equivalence relation $\equiv (\text{mod } \Psi)$ is defined as before. Let us say that $\mathfrak{M}^\circ = (U^\circ, R^\circ, S^\circ, V^\circ)$ is a *filtration* of the canonical model $\mathfrak{M}_L$ through Ψ if the following conditions are satisfied:

(i) U° is the class U/Ψ of equivalence classes x° where $x \in U_L$,
(iiA) if $(x,y) \in R_L$, then $(x^\circ, y^\circ) \in R^\circ$,
(iiB) if $(x^\circ, y^\circ) \in R^\circ$, then $[\cdot]A \in x \cap \Psi$ only if $A \in y$,
(iiiA) if $(x,y) \in S_L$, then $(x^\circ, y^\circ) \in S^\circ$,

(iiiB) if $(x^\circ, y^\circ) \in S^\circ$, then $[*]A \in x \cap \Psi$ only if $A \in y$,

(iv) if P is a propositional letter in Ψ, then $V^\circ(P) = \{x^\circ : x \in V_L(P)\}$.

Filtration Theorem (first version). For all formulæ $A \in \Psi$ and for all $x \in U_L$,

$$\mathfrak{M}^\circ \models_{x^\circ} A \text{ if and only if } \mathfrak{M}_L \models_x A.$$

Filtration Theorem (second version). For all formulæ $A \in \Psi$ and for all $x \in U_L$,

$$\mathfrak{M}^\circ \models_{x^\circ} A \text{ if and only if } A \in x.$$

Proof. The difference between the two versions is that in the former we proceed via the canonical model for L, whereas in the latter we prove the theorem from scratch. The difference is not great, but it is worth comparing the two alternatives.

The former version is proved as in the case of modal logic. The induction is on the complexity of A. Let us scrutinize the subcase when A is of the form $[\cdot]B$, where the result to be proved is assumed to hold for B (the case when A is of the form $[*]B$ is similar). Here one has to argue that the following conditions are logically equivalent:

(1) $\mathfrak{M}^\circ \models_{x^\circ} [\cdot]B$,

(2) $\forall y\, ((x^\circ, y^\circ) \in R^\circ \Rightarrow \mathfrak{M}^\circ \models_{y^\circ} B)$,

(3) $\forall y\, ((x^\circ, y^\circ) \in R^\circ \Rightarrow \mathfrak{M}_L \models_y B)$,

(4) $\mathfrak{M}_L \models_x [\cdot]B$.

The equivalence of (1) and (2) follows from the truth-condition for $[\cdot]$, that of (2) and (3) from the induction hypothesis. To go from (3) to (4), assume that $(x,y) \in R_L$, for any $y \in U_L$. Then by condition (iiA), $(x^\circ, y^\circ) \in R^\circ$, and hence $\mathfrak{M}_L \models_y B$, by (3). Consequently, $\mathfrak{M}_L \models_x [\cdot]B$. Conversely, to go from (4) to (3), assume that $\mathfrak{M}_L \models_x [\cdot]B$. Then by the Canonical Model Theorem, $[\cdot]B \in x$. Suppose that $(x^\circ, y^\circ) \in R^\circ$. Then, since $[\cdot]B \in \Psi$, condition (iiB) yields $B \in y$. Hence by the Canonical Model Theorem $\mathfrak{M}_L \models_y B$, as we wanted.

Now let us compare this passage with the corresponding passage in the proof of the second version. Here there are the following corresponding four conditions:

(1') $\mathfrak{M}^\circ \models_{x^\circ} [\cdot]B$,

(2') $\forall y\ ((x^\circ,y^\circ) \in R^\circ \Rightarrow \mathfrak{M}^\circ \models_{y^\circ} B)$,

(3') $\forall y\ ((x^\circ,y^\circ) \in R^\circ \Rightarrow B \in y)$,

(4') $[\cdot]B \in x$.

The equivalence of (1') and (2') and of (2') and (3') follows as before. Going from (4') to (3') is straightforward, thanks to condition (iiB). But going from (3') to (4') involves a certain subtlety.

Assume that (3') holds. With the help if condition (iiA), we see that $B \in y$ for all y such that $(x,y) \in R_L$. Using the definition of R_L, we note that $\{C : [\cdot]C \in x\} \vdash_L B$. Appealing to the finitariness of L and to Scott's Rule, we conclude that $\{[\cdot]C : [\cdot]C \in x\} \vdash_L [\cdot]B$; in other words, $x \vdash_L [\cdot]B$. Hence (4'), $[\cdot]B \in x$, as we wanted.

The argument just given holds a certain familiarity: we met it in the proof of the Canonical Model Theorem, in the modal part of the inductive step. The point we wish to make here is that if we prove the Filtration Theorem from scratch—the second version—dispensing with any appeal to the Canonical Model Theorem, then we still have to go over what is the crucial part of the proof of the Canonical Model Theorem. Thus the work to be done is pretty much the same in the two cases.

The analysis that now follows may be viewed as a generalization of the analysis of S4 in section 1.2.6. Keeping L and Ψ as specified we define $\mathfrak{M}\dagger = (U^\circ, R\dagger, S\dagger, V^\circ)$ as follows:

$$R\dagger = \{(x,y) : \exists x' \equiv x\ \exists y' \equiv y\ (x',y') \in R_L\},$$
$$S\dagger = (R\dagger)^*.$$

Thus it is clear from the outset that $\mathfrak{M}\dagger$ is a finite standard model. We will now show that it is a filtration. That condition (iiA) is satisfied is clear. To see that (iiB) is satisfied, suppose that $(x^\circ,y^\circ) \in R^\circ$ and that $[\cdot]A \in x \cap \Psi$. By definition there are some $x' \equiv x$ and $y' \equiv y$ such that $(x,y) \in R_L$. Since $[\cdot]A \in x \cap \Psi$, also $[\cdot]A \in x'$. Hence $A \in y'$, and since Ψ is closed under subformulæ and thus $A \in \Psi$, $A \in y$. Thus the difficulty in proving that $\mathfrak{M}\dagger$ is a filtration consists in showing that conditions (iiiA) and (iiiB) hold.

Lemma A. If $(x,y) \in S_L$, then $(x°,y°) \in S°$.

Proof. Falling back on an argument in section 1.2.5, we make the following observation: for every set $V \subseteq U°$ there is a Boolean combination C_V of formulæ in Ψ—a disjunction of conjunctions of formulæ, each of which is either a formula in Ψ or else is the negation of a formula in Ψ—such that, for all $w \in U_L$, $w° \in V$ if and only if $C_V \in w$. (This is a notion of separability subtly different from that in section 1.2.5. However, the proof of the new claim is analogous to that of the old result.)

Suppose now that $(x,y) \in S_L$. Let W be the set $\{(z° : (x°,z°) \in S\dagger\}$. By the observation just made there exists a Boolean combination $C_W = C$ of formulæ in Ψ such that, for all $t \in U_L$,

(0) $t° \in W$ if and only if $C \in t$.

Since $S\dagger$ is defined as the ancestral of $R\dagger$, we certainly have $x° \in W$, so

(1) $C \in x$.

Let u be any element in U_L such that $(x,u) \in S_L$. Suppose that $C \in u$. Then by (0) we have $u° \in W$. Hence if v is an element of U_L such that $(u,v) \in R_L$ it follows by the definition of $R\dagger$ that $(u°,v°) \in R\dagger$; therefore also $v° \in W$ and so, by (0), $C \in v$. This goes to show that $[\cdot]C \in u$. In other words, we have shown that

(2) $[*](C \supset [\cdot]C) \in x$.

But L is a normal ancestral logic, so every instance of the schema (∗ind) is in x. Hence, by (1) and (2), $[*]C \in x$. But $(x,y) \in S_L$ by assumption, so $C \in y$. Hence, by a final application of (0), $y° \in W$, which is the same as saying that $(x°,y°) \in S\dagger$. ■

Lemma B. If $(x°,y°) \in S\dagger$, then $[*]A \in x \cap \Psi$ only if $A \in y$,

Proof. We begin by proving the following claim, for all elements u, v $\in U_L$ and for all n:

(¶) If $(u^\circ, v^\circ) \in (R\dagger)^n$, then $[*]A \in u \cap \Psi$ only if $[*]A \in v$.

The claim is proved by induction on n. If $(u^\circ, v^\circ) \in (R\dagger)^0$ then $u^\circ = v^\circ$, and so $u \equiv v$. Thus in this case it is trivial that $[*]A \in v$ if $[*]A \in u \cap \Psi$. Assume therefore that the claim (¶) holds for n. Suppose that $(u^\circ, v^\circ) \in (R\dagger)^{n+1}$ and that $[*]A \in u \cap \Psi$. Then there is some $w \in U_L$ such that

(1) $(u^\circ, w^\circ) \in R\dagger$,

(2) $(w^\circ, v^\circ) \in (R\dagger)^n$.

Condition (1) implies the existence of u and v such that $u' \equiv u$ and $w' \equiv w$ and $(u', w') \in R_L$. Evidently, $[*]A \in u'$. Above we have seen that $[*]A \supset [\cdot][*]A$ is a thesis of L. Hence $[\cdot][*]A \in u'$ and so $[*]A \in w'$. Since Ψ is closed under subformulæ, $[*]A \in w$. This result places us in a position to apply the induction hypothesis to w and v; the conclusion is that $[*]A \in v$. This ends the proof of (¶).

Returning to the main proof, suppose that $(x^\circ, y^\circ) \in S\dagger$ and $[*]A \in x \cap \Psi$. As $\mathfrak{M}\dagger$ is standard there is some n such that $(x^\circ, y^\circ) \in (R\dagger)^n$. By the observation (¶), therefore, $[*]A \in y$. But $[*]A \supset A$ is a thesis of L and so is an element of y. Hence $A \in y$.

Corollary. $\mathfrak{M}\dagger$ is a filtration.

Theorem. The axiom system $\mathfrak{A}$ is sound and complete with respect to the class of all ancestral frames. In other words, the basic ancestral logic is axiomatized by $\mathfrak{A}$.

Proof. Let Σ be a finite set of formulæ consistent in $\Lambda(\mathfrak{A})$, the logic generated by the axiom system $\mathfrak{A}$. By Lindenbaum's Lemma there is some maximal consistent set x extending Σ. Let Ψ be the set of all subformulæ of formulæ in Σ. Then Ψ is a finite set, and we can construct the model $\mathfrak{M}\dagger$ as described above. By the corollary, $\mathfrak{M}\dagger$ is a filtration of the canonical model $\mathfrak{M}\dagger$ for $\Lambda(\mathfrak{A})$ through Ψ. By the filtration theorem, therefore, Σ is satisfied at x° in $\mathfrak{M}\dagger$. And the frame of $\mathfrak{M}\dagger$ is a frame for $\Lambda(\mathfrak{A})$.

The basic ancestral logic is thus an example of a logic that is finitary without being compact.

Corollary. The basic ancestral logic has the finite model property.

Notice that even though what we have proved is a weak completeness result—noncompactness precludes strong completeneness—decidability still follows.

Theorem. The basic ancestral logic is decidable.

It is always a good idea to review a long proof and try to pinpoint the rôles of the various assumptions that have been made. In the case of $\mathfrak{A}$ the following remarks apply.

> $\Lambda(\mathfrak{A})$ is a finitary logic normal in both [·] and [∗]. There is a crucial point in the proof of the Canonical Model Theorem that depends on these features. As we saw, if one prefers to sidestep canonical models, the Filtration Theorem takes a form that requires the same crucial point to be proved.
>
> Without finitariness Lindenbaum's Lemma could not be invoked in the completeness proof.
>
> $\Lambda(\mathfrak{A})$ provides (∗ind). Without it we would not have been able to prove Lemma A.
>
> $\Lambda(\mathfrak{A})$ provides $[*]A \supset A$ and $[*]A \supset [\cdot][*]A$. Without them we would not have been able to prove Lemma B.

3. Dynamic logic

3.1. Semantics

1. Frames

Semantically speaking, in modal logic we have one accessibility relation, in ancestral logic two. In dynamic logic we go the whole way: there are indefinitely many accessibility relations. This time, though, the intuition is somewhat different. In modal (deontic, epistemic) logic there is a static universe; from a point one may have access to other points, but no change is envisaged. Ancestral logic is completely abstract and thus as static as modal logic. In tense-logic time is something that happens to you, not anything that is up to you. In dynamic logic the idea is that the accessibility relations are actions. This way of representing actions is certainly rudimentary, but it is a beginning. Dynamic logic is a logic of action of a primitive kind.

Let U be any set. By an *action* in U we understand any binary relation in U. Then by a *frame* we may understand a pair (U, R) where U is a set (the *universe* of the frame) and R is a family of actions (the *repertoire* of the frame).

Such a general concept of frame is not very interesting, though. Let us say that a frame (U, R) is *standard* if R satisfies the following conditions:

> if $\mathbf{a}, \mathbf{b} \in R$ then $\mathbf{a} \cup \mathbf{b} \in R$ and $\mathbf{a} \mid \mathbf{b} \in R$,
> if $\mathbf{a} \in R$ then $\mathbf{a}^* \in R$,
> if $X \subseteq U$ then *test* $X \in R$.

Here $\cup$ stands for set theoretical union, $\mid$ for relative product, and * for the ancestral. Furthermore, *test* $X = \Delta_U \upharpoonright X$, where Δ_U is the diagonal relation in U, that is, $\{(x,x) : x \in U\}$, and $\upharpoonright$ indicates restriction to X.

A more general description of a frame would be as a structure (U, R, P), where U is a set, R = {|R|, ∪, |, *, test} is an algebra of relations in U ("actions") and P = {|P|, ∩, ∪, –, 0, 1, after} is an algebra of subsets of U ("propositions"). Here |R| and |P| are the carriers of the respective algebras, and the operator after is defined by the condition after($\mathbf{a}$,X) =

$\{x : \forall u\ ((x,u) \in \mathbf{a} \Rightarrow u \in X\}$. The latter is obviously related to the "interior" operators discussed in the section on algebraic semantics in chapter 1. Then standard frames would be those where R is a regular algebra (meaning that the condition on the operations of R listed above are satisfied) and P is a normal modal algebra such that $P = \mathfrak{P}U$. However, we shall not pursue this line in the present notes.

Central among the indefinitely many operations in a frame are the members of a family which we will now describe. There are two basic semantic categories in our theory, that of propositions and that of actions. If U is a universe, then let P be the set of propositions, R the set of actions. (In our theory, P is simply the power set $\mathfrak{P}U$ of U.) The family we have in mind is the class of all operations belonging to one of the following two types:

$$P^m \times R^n \longrightarrow P,$$
$$P^m \times R^n \longrightarrow R.$$

In the former case we say that the operation is *proposition-forming* (yielding a proposition if applied to m propositions and n actions), in the latter that it is *action-forming* (yielding an action when applied to m propositions and n actions).

We should like to find the logic determined by the new concept of frame. To do so we must first decide on an object language suited to this kind of frame.

2. Language

What would be a fruitful object language for reasoning about standard frames? The answer to this question depends on what model theoretical operations that are thought to be interesting. To parallel the structure of our semantical machinery, let us postulate two syntactic categories F (formulæ) and T (terms). In each category there are denumerably many primitive symbols, all distinct (*propositional letters* and *action letters*). Every other primitive symbol belongs in either of the following categories:

$$F^m \times T^n \longrightarrow F,$$
$$F^m \times T^n \longrightarrow T.$$

The former are *formula makers*, the latter *term makers*. The formula makers are (i) Boolean operators, of type $F^m \longrightarrow F$, for some m, (ii) the higher order operator [ ] ("after") of type $F \times T \longrightarrow F$. The term makers are + ("sum") and ; ("composition") of type $T^2 \longrightarrow T$, * ("the Kleene star") of type $T \longrightarrow T$, and ? ("test") of type $F \longrightarrow T$.

A more traditional definition of the language would be the following inductive definition, which defines 'formula' and 'term' at the same time:

1. Every propositonal letter is a *formula.*
2. Every action letter is a *term.*
3. If o is an n-ary Boolean operator and $A_0, ..., A_{n-1}$ are *formulæ*, then $o(A_0, ..., A_{n-1})$ is a *formula.*
4. If A is a *formula* and α is a *term*, then $[\alpha]A$ is a *formula.*
5. If α and β are *terms*, then $\alpha + \beta$ is a *term.*
6. If α and β are *terms*, then $\alpha;\beta$ is a *term.*
7. If α is a *term*, then $\alpha*$ is a *term.*
8. If A is a *formula*, then ?A is a *term.*
9. Nothing is a *formula* or a *term* except by virtue of 1 - 8.

We say that an expression is *well-formed* if and only if it is either a formula or a term.

This object language was first defined by Vaughan Pratt who wanted to use it in order to discuss, in a formalized way, the effect of programs. Informally, a command to do $\alpha + \beta$ is carried out by doing either α or β (it does not matter which); a command to do $\alpha;\beta$ is carried out by first doing α, then doing β; a command to do $\alpha*$ is carried out by doing α some finite number of times (0 or 1 or 2 or...—it does not matter which). The command ?A is carried out by verifying that A obtains. If A does not obtain, then it is obviously impossible to verify that A obtains. For this reason, calling ?A a test program, which is often done, is slightly misleading: the label "test" may suggest that one is expecting a yes-or-no answer, but that is not the case.

3. Models

Only now can we define the notion of a model. Let us say that V is a *valuation* in a frame (U, R) if it is a function assigning a subset of U to each propositional letter and an action in R to each action letter. A *model* is a structure (U, R, V) where (U, R) is a frame and V is a valuation in (U, R). We define the *meaning* or *intension* $\|E\|^{\mathfrak{M}}$ of well-formed expressions E in a given model $\mathfrak{M}$ = (U, R, V) as follows (although for convenience we shall omit the superscript).

1. For every propositional letter $\|P\| = V(P)$.
2. For every action letter π, $\|\pi\| = V(\pi)$.
3. For Boolean operators the conditions are obvious. For example, if $\wedge$ is primitive, then $\|A \wedge B\| = \|A\| \cap \|B\|$; if $\neg$ is primitive, then $\|\neg A\| = U - \|A\|$; etc.
4. $\|[\alpha]A\| =$ after $(\|\alpha\|, \|A\|)$.
5. $\|\alpha + \beta\| = \|\alpha\| \cup \|\beta\|$.
6. $\|\alpha;\beta\| = \|\alpha\| \mid \|\beta\|$.
7. $\|\alpha*\| = \|\alpha\|^*$.
8. $\|?A\| =$ test $\|A\|$.

(Here *after* is as in section 1.) Notice that this definition parallels, step by step, the definition of well-formed expressions. Notice also:

the intension of a formula is a proposition,
the intension of a term is an action.

We say that A is *true at* x (in $\mathfrak{M}$) if $x \in \|A\|$. Other semantical notions, including those involving truth and validity, are taken over from modal logic. In particular we can now pose a completeness problem: how to axiomatize the *basic propositional dynamic logic* (PDL), that is, the set of formulæ valid in every standard frame?

3.2. Syntax

1. Axiom systems

Let $\mathfrak{D}$ be the axiom system whose inference rules are (MP) and (RS) for every operator $[\alpha]$, and whose axioms are the instances of the following schemata:

(+)	$[\alpha + \beta]C \equiv ([\alpha]C \wedge [\beta]C)$,
(;)	$[\alpha;\beta]C \equiv [\alpha][\beta]C$,
(*T)	$[\alpha*]C \supset C$,
(*E_1)	$[\alpha*]C \supset [\alpha]C$,,
(*4)	$[\alpha*]C \supset [\alpha*][\alpha*]C$,
(*ind)	$(C \wedge [\alpha*](C \supset [\alpha]C)) \supset [\alpha*]C$,
(?)	$[?A]C \equiv (A \supset C)$.

In the light of the discussion in chapter 2 one realizes that this is only one of a number of axiom systems, all of which lead to the same class of formal theorems.

Soundness Theorem. All theorems of $\mathfrak{D}$ are valid in every standard frame.

A *dynamic* logic is defined as a logic containing as theses all theorems of $\mathfrak{D}$. A dynamic logic is *normal* if it is closed under (RC), for every $[\alpha]$. (or equivalently under (RN) or (RS) or (RE) or (RPE)). Thus in a normal dynamic logic every operator $[\alpha]$ is at least a K-operator, and every operator $[\alpha*]$ is at least an S4-operator.

2. Fischer/Ladner closure

By the *Fischer/Ladner conditions* we mean the following:

(FL0)	If o is any n-ary formula making operator, then $o(A_0, ..., A_{n-1}) \in \Psi$ only if $A_0, ..., A_{n-1} \in \Psi$.
(FL+)	If $[\alpha + \beta]C \in \Psi$ then $[\alpha]C \in \Psi$ and $[\beta]C \in \Psi$.
(FL;)	If $[\alpha;\beta]C \in \Psi$ then $[\alpha][\beta]C \in \Psi$.
(FL*)	If $[\alpha*]C \in \Psi$ then $[\alpha][\alpha*]C \in \Psi$.
(FL?)	If $[?A]C \in \Psi$ then $A \in \Psi$.

The *Fischer/Ladner closure* of a set Ψ is the smallest set closed under the Fischer/Ladner conditions that includes Ψ.

The Fischer/Ladner Lemma. The Fischer/Ladner closure of a finite set is finite.

Proof. It seems difficult to give a rigorous proof of this result that is also reasonably intelligible. Here we shall sacrifice rigour in the hope of maintaining intelligibility. (Readers dissatisfied with the lack of rigour are invited to work out a rigorous proof based on the outline we present here.)

We show how to construct, for each formula in the language, a certain tree. Every node is of the form (**n**, A), where **n** is a label and A the associated formula. We make an ad hoc distinction between formulæ that are "underlined" and formulæ that are "not underlined", a distinction not made in the statement of the Fischer/Ladner conditions. Assume, for the sake of example, that conjunction and negation are our primitive Boolean operators. The following inductive definition contains instructions for how to grow our trees. Unlike trees you see in nature ours grow downwards.

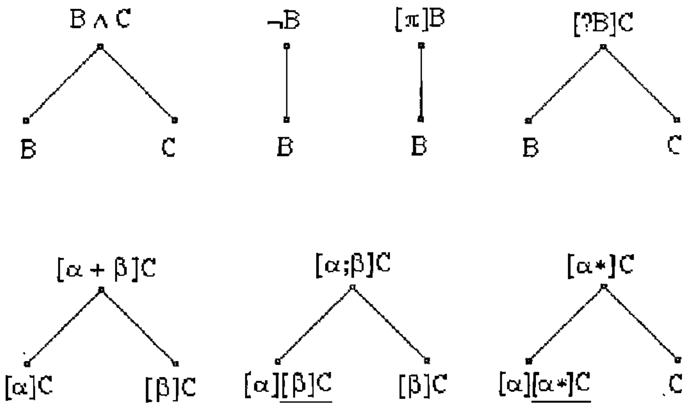
1. If (**n**, P) is a node in the tree, where P is a propositional letter, then nothing is to be added under that node.

2. If (**n**, A) is a node where the associated formula A is completely underlined, then nothing is to be added under that node.

3. If (**n**, A) is a node where the associated formula A is not completely underlined, then proceed as follows. Let B and C stand for formulæ that may or may not be underlined, completely or partially.
   (i) If A is $B \wedge C$ then find new labels **n**' and **n**" and add (**n**', B) and (**n**", C) as new nodes directly under (**n**, A).
   (ii) If A is $\neg B$, then find a new label **n**' and add (**n**', B) as a new node directly under (**n**, A).
   (iii) If A is $[\pi]B$, then find a new label **n**' and add (**n**', B) as a new node directly under (**n**, A).
   (iv) If A is $[\alpha + \beta]C$, then find new labels **n**' and **n**" and add (**n**', $[\alpha]C$) and (**n**", $[\beta]C$) as new nodes directly under (**n**, A).
   (v) If A is $[\alpha;\beta]C$, then find new labels **n**' and **n**" and add (**n**', $[\alpha]\underline{[\beta]C}$) and (**n**", $[\beta]C$) as new nodes directly under (**n**, A). Notice that part of the formula associated with **n**' is underlined.
   (vi) If A is $[\alpha*]C$, then find new labels **n**' and **n**" and add (**n**', $[\alpha]\underline{[\alpha*]C}$) and (**n**", C) as new nodes directly under (**n**, A). Notice that part of the formula associated with **n**' is underlined.
   (vii) If A is $[?B]C$ then find new labels **n**' and **n**" and add (**n**', B) and (**n**", C) as new nodes directly under (**n**, A).

One gets a better grasp of these instructions by depicting them graphically, as in the chart on the next page.

Let $\mathfrak{T}$ be the tree generated by any formula A, and let $\Psi(\mathfrak{T})$ be the set of formulæ (disregarding any underlining) that are associated with at least some node in $\mathfrak{T}$. Notice that every underlined formula or subformula of an associated formula in the tree appears somewhere else without underlining. Hence every formula of the Fischer/Ladner closure of the set {A} is an element of $\Psi(\mathfrak{T})$. Also the converse holds, so $\Psi(\mathfrak{T})$ is in fact exactly the Fischer/Ladner closure of {A}.



It is clear that every branch in $\mathfrak{T}$ terminates. This is so because any formula that is added is simpler than the preceding formula in every case except when $[\alpha]\underline{[\beta]C}$ is added under $[\alpha;\beta]C$, or $[\alpha]\underline{[\alpha^*]C}$ is added under $[\alpha*]C$. But underlined formulæ will not influence later growth of the tree: the only part of $[\alpha]\underline{[\beta]C}$ or $[\alpha]\underline{[\alpha^*]C}$ that will give rise to growth is the initial $[\alpha]$-operator. For our purposes the complexity of $[\alpha]\underline{[\beta]C}$ and $[\alpha]\underline{[\alpha^*]C}$ is therefore the same as $[\alpha]Q$, where Q is a propositional letter. Hence also in these two cases are the successor formulæ simpler than the predecessors.

Thus $\mathfrak{T}$ is a finitely-branching tree, every branch of which is finite. By König's Lemma, such trees contain at most finitely many nodes. Hence $\Psi(\mathfrak{T})$ is finite. This proves the lemma, for the Fischer/Ladner closure of any set Σ is the union of the Fischer/Ladner closures of the sets $\{A\}$ with $A \in \Sigma$. ■

3.3. *Completeness*

1. *Canonical models*

Let L be finitary normal dynamic logic. Also in this case can we define *the canonical model* $\mathfrak{M}_L = (U_L, R_L, V_L)$ *for L. To do so, just* generalize the modal concept in the obvious way: define

U_L = the set of maximal L-consistent sets,
R_L = the family of all relations $R_L(\alpha)$ where α is a term in the language and

$$R_L(\alpha) = \{(x,y) : \forall C\, ([\alpha]C \in x \Rightarrow C \in y\},$$

$V_L(P) = \{x : P \in x\}$, for every propositional letter P,
$V_L(\pi) = R_L(\pi)$, for every action letter π.

The Canonical Model Theorem can be proved, so $\mathfrak{M}_L$ is indeed a model for L.

However, $\mathfrak{M}_L$ is not a standard model (the main "defect" is that $(R(\alpha))^*$ is in general only a proper subset of $R(\alpha*)$). In order to achieve a completeness result we would have to transform $\mathfrak{M}_L$ into a standard model without changing whatever truth-conditions are dear to us. One technique for doing this is by way of filtration. This is the way we went when faced with the corresponding problem in ancestral logic. However, as we remarked there, the detour via the canonical model is not really necessary. Here we shall go directly for the filtration.

2. Filtrations

Let Ψ be a finite set of formulæ closed under subformulæ. We write $\alpha\ \eta\ \Psi$ if α is a term occurring in some formula in Ψ. As before we designate by $\equiv \pmod{\Psi}$ the equivalence relation induced by Ψ in U_L, writing x° for the equivalence class $\{x' \in U_L : x \equiv x' \pmod{\Psi}\}$. Let us write U° for the set U/Ψ of equivalence classes in U_L. We say that a model $\mathfrak{M}^\circ = (U^\circ, R^\circ, V^\circ)$ is a *filtration* through Ψ if

(A) if $(x,y) \in R_L(\alpha)$ then $(x^\circ, y^\circ) \in R^\circ(\alpha)$,
(B) if $(x^\circ, y^\circ) \in R^\circ(\alpha)$ then $[\alpha]A \in x \cap \Psi$ only if $A \in y$,
(C) $V^\circ(P) = \{x^\circ : P \in x\}$ for every propositional letter $P \in \Psi$.

We mark intensions in $\mathfrak{M}^\circ$ by the little ring. For each formula $A \in \Psi$ let us write $|A|^\circ = \{x^\circ : A \in x\}$. The following is an immediate generalization of the filtration theorem of modal logic:

Theorem. For every formula $A \in \Psi$, $\|A\|^\circ = |A|^\circ$. In other words, for every formula $A \in \Psi$ and every point $x \in U_L$,

$$\mathfrak{M}^\circ \models_{x^\circ} A \text{ if and only if } A \in x.$$

Proof. See the proof of the filtration theorem for ancestral logic, the second version.

Let Ψ be a given finite set of formulæ but this time closed under the Fischer/Ladner conditions. We shall construct a particular model with universe U°, where as before U° is the class U/Ψ of equivalence classes in U_L of $\equiv$ (mod Ψ). We define a family of binary relations $|\alpha|\dagger$ in U/Ψ as follows.

$\langle x', y' \rangle$

$$|\pi|\dagger = \{(x^\circ,y^\circ) : \exists x' \equiv x\ \exists y' \equiv y\ (x,y) \in R_L(\pi)\},\ \text{for every action letter } \pi\ \eta\ \Psi.$$

$$|\alpha + \beta|\dagger = |\alpha|\dagger \cup |\beta|\dagger,$$

$$|\alpha;\beta|\dagger = |\alpha|\dagger \mid |\beta|\dagger,$$

$$|\alpha*|\dagger = (|\alpha|\dagger)^*,$$

$$|?A|\dagger = \text{test } |A|^\circ.$$

Now define $\mathfrak{M}\dagger = (U/\Psi,\ R\dagger,\ V\dagger)$,

$$R\dagger = \{|\alpha|\dagger : \alpha\ \eta\ \Psi\},$$

$$V\dagger(P) \begin{cases} = |P|^\circ, & \text{for every propositional letter } P \in \Psi, \\ = \emptyset, & \text{for every other propositional letter } P. \end{cases}$$

$$V\dagger(\pi) \begin{cases} = |\pi|\dagger, & \text{for every action letter } \pi\ \eta\ \Psi, \\ = \emptyset, & \text{for every other action letter } \pi. \end{cases}$$

Notice that $\mathfrak{M}\dagger$ is a standard model. We shall now prove that, given that L is a finitary normal dynamic logic, $\mathfrak{M}\dagger$ is a filtration.

Lemma A. If $(x,y) \in R_L(\gamma)$, then $(x^\circ,y^\circ) \in |\gamma|\dagger$, for all $\gamma\ \eta\ \Psi$.

Proof. By induction on γ. Suppose that $(x,y) \in R_L(\gamma)$. If γ is an action letter π in Ψ, then the claimed result follows by the way $|\pi|\dagger$ was defined.

If $\gamma = ?C$ for some formula C then suppose that $A \in x$. Then $C \supset A \in x$, so by the "right-to-left" half of the axiom schema called (?) $[?C]A \in x$. Hence $A \in y$. This shows that $x \subseteq y$ and hence that $x = y$. Since $C \in \Psi$, $|C|^\circ$ is well-defined. Consequently, $(x^\circ,y^\circ) \in \Delta^\circ 1\ |C|^\circ$, which is to say that $(x^\circ,y^\circ) \in |?C|\dagger$.

Suppose now that the result holds for some terms α and β in Ψ. We must check the cases when γ is $\alpha + \beta$ or $\alpha;\beta$ or $\alpha*$.

First suppose that $\gamma = \alpha + \beta$. We contend that $(x,y) \in R_L(\alpha)$ or $(x,y) \in R_L(\beta)$. For suppose not. Then there are formulæ A and B such that $[\alpha]A \in x$ and $[\beta]B \in x$ while $A \notin y$ and $B \notin y$. By modal logic, $[\alpha](A \vee B) \in x$ and $[\beta](A \vee B) \in x$. Hence by axiom schema (+) ("right-to-left"), $[\alpha + \beta](A \vee B) \in x$. Then $A \vee B \in y$, a contradiction; which ends the proof of our contention. By the induction hypothesis, $(x^\circ,y^\circ) \in |\alpha|\dagger$ or $(x^\circ,y^\circ) \in |\beta|\dagger$. In either case, $(x^\circ,y^\circ) \in |\alpha + \beta|\dagger$ by definition of $\mathfrak{M}\dagger$.

Next suppose that $\gamma = \alpha;\beta$. We contend that there is an element w such that $(x,w) \in R_L(\alpha)$ and $(w,y) \in R_L(\beta)$. Consider the set $\Sigma = \{A : [\alpha]A \in x\} \cup \{\langle\beta\rangle B : B \in y\}$. If Σ were L-inconsistent, then there would be some $A_0, \ldots, A_{m-1}$ and $B_0, \ldots, B_{n-1}$ such that $[\alpha]A_0, \ldots, [\alpha]A_{m-1} \in x$ and $B_0, \ldots, B_{n-1} \in y$ and

$$\vdash_L (A_0 \wedge \ldots \wedge A_{m-1} \wedge \langle\beta\rangle B_0 \wedge \ldots \wedge \langle\beta\rangle B_{n-1}) \supset \bot$$

By modal logic, therefore, $(A_0 \wedge \ldots \wedge A_{m-1} \wedge \langle\beta\rangle(B_0 \wedge \ldots \wedge B_{n-1})) \supset \bot$ is a thesis of L, and hence

$$\vdash_L (A_0 \wedge \ldots \wedge A_{m-1}) \supset [\beta]\neg(B_0 \wedge \ldots \wedge B_{n-1}).$$

By Scott's Rule,

$$\vdash_L ([\alpha]A_0 \wedge \ldots \wedge [\alpha]A_{m-1}) \supset [\alpha][\beta]\neg(B_0 \wedge \ldots \wedge B_{n-1}),$$

and so finally, by the schema (;) ("right-to-left"),

$$\vdash_L ([\alpha]A_0 \wedge \ldots \wedge [\alpha]A_{m-1}) \supset [\alpha;\beta]\neg(B_0 \wedge \ldots \wedge B_{n-1}).$$

Evidently then $[\alpha;\beta]\neg(B_0 \wedge \ldots \wedge B_{n-1}) \in x$, therefore $\neg(B_0 \wedge \ldots \wedge B_{n-1}) \in y$, a contradiction. This argument shows that Σ is indeed L-consistent. Hence, by Lindenbaum's Lemma, there is some maximal L-consistent extension w of Σ. It is clear that w has the right properties, so our contention has now been proved. By the induction hypothesis, $(x^\circ,w^\circ) \in |\alpha|\dagger$ and $(w^\circ,y^\circ) \in |\beta|\dagger$. Hence $(x^\circ,y^\circ) \in |\alpha;\beta|\dagger$ by the definition of $\mathfrak{M}\dagger$.

Finally suppose that $\gamma = \alpha*$. This is the most intricate link in proof of the lemma. However, it is completely analogous to the proof of

Lemma A for ancestral logic, so we omit the details. Suffice it to say that it is in this part that the "induction schema" (*ind) is needed, and that it matters that Ψ is finite. ∎

Lemma B. If $(x^\circ, y^\circ) \in |\gamma|\dagger$ then $[\gamma]C \in x \cap \Psi$ only if $C \in y$.

Proof. By induction on γ. Assume that $(x^\circ, y^\circ) \in |\gamma|\dagger$ and $[\gamma]C \in x \cap \Psi$. First suppose that γ is an action letter π. Then there are elements $x' \equiv x$ and $y' \equiv y$ such that $(x', y') \in R_L(\pi)$. The fact that $[\pi]C \in x \cap \Psi$ implies that $[\pi]C \in x'$, hence that $C \in y'$. But Ψ is closed under subformulæ (among many other conditions), so $C \in \Psi$. Hence $C \in y$.

Next suppose that $\gamma = ?A$, for some formula A; note that by (FL0) and (FL?) both $A \in \Psi$ and $C \in \Psi$. Then $x^\circ = y^\circ$ and $A \in x$. By axiom schema (?) ("left-to-right"), $A \supset C \in x$, hence $C \in x$. Since $x \equiv y$, $C \in y$.

Suppose now that the result holds for some terms α and β in Ψ. We must check the cases when γ is $\alpha + \beta$ or $\alpha;\beta$ or $\alpha*$.

First suppose that $\gamma = \alpha + \beta$. In this case our assumption is that $[\alpha + \beta]C \in x \cap \Psi$. By axiom schema (+) ("left-to-right"), $[\alpha]C \in x$ and $[\beta]C \in x$. By (FL+), both $[\alpha]C \in \Psi$ and $[\beta]C \in \Psi$. Moreover, either $(x^\circ, y^\circ) \in |\alpha|\dagger$ or $(x^\circ, y^\circ) \in |\beta|\dagger$. The induction hypothesis, applied to whichever case obtains, gives us $C \in y$.

Next suppose that $\gamma = \alpha;\beta$. In this case our assumption is that $[\alpha;\beta]C \in x \cap \Psi$. By axiom schema (;) ("left-to-right") and (FL;),

(1) $[\alpha][\beta]C \in x \cap \Psi$.

By construction of $R\dagger$ there is some w such that

(2) $(x^\circ, w^\circ) \in |\alpha|\dagger$,

(3) $(w^\circ, y^\circ) \in |\beta|\dagger$.

The induction hypothesis used on (1) and (2) gives us $[\beta]C \in w$. By (FL0), $[\beta]C \in \Psi$. Hence

(4) $[\beta]C \in w \cap \Psi$.

The induction hypothesis used on (3) and (4) gives us $C \in y$.

Finally suppose that $\gamma = \alpha *$. This case goes through in very much the same way as the proof of Lemma B for ancestral logic, so we omit the details. Note, however, that for this step we need axiom schemata (*T), (*E_1), and (*4) as well as the condition (FL*).

Corollary. $\mathfrak{M}^\dagger$ is a filtration.

3. Completeness of PDL

Suppose that Σ is a finite $\mathfrak{D}$-consistent set of formulæ. Then, by Lindenbaum's Lemma, there is some maximal $\mathfrak{D}$-consistent set x such that $\Sigma \subseteq x$. Let Ψ be the Fischer/Ladner closure of Σ; as we saw, Ψ will be finite. Construct $\mathfrak{M}^\dagger$ as in the preceding section. Then Σ is satisfied at x° in $\mathfrak{M}^\dagger$. As we remarked before, $\mathfrak{M}^\dagger$ is a standard model. Hence every finite $\mathfrak{D}$-consistent set is satisfiable in a standard frame.

*

Readers who have followed the exposition up to this point may now go back over the last completeness proof and ask how the different pieces fit together, just as we did after the completeness proof for ancestral logic. In particular there are the following questions:

> Where does it matter that $\mathfrak{D}$ is finitary?
>
> Where does the syntactic strength of $\mathfrak{D}$ come into play—the classical Boolean part, the modal part (every operator $[\alpha]$ is normal), the axiom schemata (*T), (*E_1), (*4) and (*ind)?
>
> Why was it important that Ψ was closed under subformulæ? Under the full Fischer/Ladner conditions?

Being able to answer these questions is a sign that you have understood the long proof.

3.4. Limitations of PDL

1. Path semantics

The semantics studied in these chapters may be called *relational* semantics. In the present chapter we have represented actions (programs) semantically as relations. Is this a good representation? The question is obviously incomplete: good for what? For some purposes the terse representation of PDL is adequate, for others it may not be. Instead of trying to be more specific, let us consider an alternative.

Dynamic logic rests on the observation that an automaton is always in one total state or other and that it makes sense to associate with the automaton a space of all those total states in which it could possibly be. A particular run (computation, execution) by the automaton of a program can then be represented in this space by a *path*, namely, the sequence of total states that the automaton goes through during the run (in the order in which it goes through them). Paths evidently divide into three categories: those that halt (the computation completed), those that fail without halting (the computation not completed), and those that never stop. If we assume the automaton to work in a discrete fashion, we can associate a *signature* $(H(\alpha), F(\alpha), I(\alpha))$ with a program α, where $H(\alpha)$ is the set of halt paths (each halt path being finite), $F(\alpha)$ is the set of fail paths (each fail path also being finite), and $I(\alpha)$ is the set of infinite paths.

Here we have another way of representing a program. It goes without saying that the representation in path semantics—if we may call it so—is much richer, contains much more information than the representation in relational semantics. Given a signature $(H(\alpha), F(\alpha), I(\alpha))$ we can define a relational representation $R(\alpha)$ as the set of all pairs (x,y) for which there is some halt path from x to y. More precisely, define p as a path in U if p is a function from some initial J of the set of natural numbers into U. Obviously, J is finite if and only if $p \in H(\alpha) \cup F(\alpha)$, and J equals the set of natural numbers if and only if $p \in I(\alpha)$. There is no harm in treating the function p as a sequence of elements. Thus $p(0)$ is the first element of p, and if p is finite we shall write $p(\#)$ for it last element. We may now define

$$R(\alpha) = \{(x,y) : \exists p \in H(\alpha)\ (p(0) = x\ \&\ p(\#) - y)\}.$$

From this point of view, $R(\alpha)$ gives an exceedingly schematic representation of the program α, ignoring all fail or infinite paths and also all intermediate states in the halt paths. Thus given only a relation R there is no possibility of reconstructing the signature of α.

The fact that path semantics is much richer than relational semantics of course does not mean that is "better". Analysts always have to strike a reasonable balance between expressiveness and simplicity: the more powerful (detailed, sensitive, sophisticated) the modelling is, the less tractable the analysis tends to be. The poverty of the relational semantics is a virtue in some contexts. However, one must be clear about what it can do and what it cannot do. In a following section we shall illustrate this remark.

2. A warning regarding IF - THEN - ELSE and WHILE

One of the attractions of dynamic logic is that, in a certain sense, by its means one can formalize the important operators IF A THEN α ELSE β and α WHILE A. The sense in which it can be done is this: there are terms that can simulate the effect of these operators. More precisely, the following formulæ are valid in PDL, for all C:

$$[\text{IF } A \text{ THEN } \alpha \text{ ELSE } \beta]C \equiv [(?A;\alpha) + (\neg A;\beta)]C,$$

$$[\alpha \text{ WHILE } A]C \equiv [(?A;\alpha)*;(?\neg A)]C.$$

However, this does not mean it is desirable, let alone necessary, to identify IF A THEN α ELSE β with $(?A;\alpha) + (\neg A;\beta)$ or α WHILE A with $(?A;\alpha)*;(?\neg A)$. On the contrary, intuitively one probably feels that they are different actions; in path semantics they certainly are. To see this, just note that in any model the action $\|(?A;\alpha) + (\neg A;\beta)\|$ will always contain a fail path at every point (that is, $\langle x \rangle \in F((?A;\alpha) + (\neg A;\beta))$, for all points x in the universe), but $\|\text{IF } A \text{ THEN } \alpha \text{ ELSE } \beta\|$ need not contain any fail path. Similarly, $\|(?A;\alpha)*;(?\neg A)\|$ also always contains fail paths, but $\|\alpha \text{ WHILE } A\|$ need not do so.

There are of course many other examples of the same phenomenon; that is, of actions that cannot be distinguished in PDL even though intuition and path semantics distinguish them—the simplest example is perhaps offered by $\|?\top\|$ and $\|?\top + ?\bot\|$.

We shall now give the promised example of a case when the relational semantics does not suffice but requires some enrichment.

3. The delta operator

Some actions can be characterized as resulting in a certain state-of-affairs. Thus opening a door results in the door being open (at the moment the action has been completed); killing a mosquito results in the mosquito being dead. It might not be easy to give a full analysis of such actions, but as a first approximation one might introduce an operator δ with the idea that δA is the action consisting in bringing it about that A.

Suppose we want to pursue this idea within the context of dynamic logic. What semantic conditions would be appropriate for δ? Several decisions must be made. First there is the distinction between reliable and unreliable doings. If a mediocre darts player hits the bull's eye, then one of many descriptions of the action he just performed is that it consisted in hitting the bull's eye. But (under normal circumstances) his success was by no means assured. If he tries to repeat his action (by running "the same program" a second time) he may well fail. This is an example of unreliable doing. The analysis of such doing seems more difficult than that of reliable doing. Hence our decision to restrict δ to reliable doing.

Next we must face the fact that often there are several ways of bringing about one and the same state-of-affairs. Rather than choosing between them or trying to impose some kind of ordering on them (with a view to designating some of them as "normal" ways of performing the action) we go for maximality and recognize them all: given a frame we define the intension of δA as the set of pairs (x,y) such that for some action $\mathbf{a}$ in the repertoire, $\mathbf{a}$ is a reliable way of seeing to it that A is true at y, and $(x,y) \in \mathbf{a}$. Formally, if (U, R) is a given frame,

$$\|\delta A\| = \{(x,y) : \exists \mathbf{a} \in R\ ((x,y) \in \mathbf{a}\ \&\ \forall z\ ((x,z) \in \mathbf{a} \Rightarrow z \in \|A\|)\}.$$

This, then, defines the delta of ***maximal, reliable*** doing. There are other ways of defining delta, perhaps more interesting. Still, this is one possibility, and it has some claim to interest.

However (this is the point of the example!), the definition just given is not in accord with our intuitions as described. To see this, suppose that α names an action that, at a particular point x in a model, is an unreliable way of seeing to it that A. In other words, there are points y and w such that $(x,y) \in \|A\|$ and $(x,w) \notin \|A\|$. Then $\|\alpha;?A\| \subseteq \|\delta A\|$. That is to say, on our definition, a reliable way of doing A is to do anything and then ask whether A obtains; if it does, we have achieved A, if not the run has failed and so does not count. Thus $(x,y) \in \|\alpha;?A\|$ since $(x,y) \in \|\alpha\|$ and $(y,y) \in \|?A\|$, but $(x,w) \notin \|\alpha;?A\|$ since, even though $(x,w) \in \|\alpha\|$, still $(w,w) \notin \|?A\|$.

Our formal result is of course informally absurd: no-one would wish to claim that, in general, $\alpha;?A$ is a reliable way of seeing to it that A. The mistake in the formal analysis sketched above was to try to carry out within the relational semantics of PDL a project that evidently requires greater resources. In particular, for the delta operator it is not enough to consider just runs that terminate: if we do not wish to adopt path semantics in all its rich complexity, at least we must find some other way to register the possibility that paths may fail or be infinite. [Readers interested in a further discussion of these matters are referred to the author's article "Action incompleteness" in *Studia logica*, vol. 51 (1992).]

4. Background

4.1. Historical remarks

1. Modal logic. Philosophers have been interested in modal notions—necessity, possibility, contingency since Aristotle, and some, for example Aristotle himself, have tried to study their logic. Modern modal logic may be said to have begun round 1912 when C. I. Lewis, upon reading Russell and Whitehead's *Principia mathematicæ*, became interested in trying to find a connective more suited than material implication to express our informal concept of entailment. Thanks to Lewis and others a formalism for "alethic" modal logic was developped. With time, logicians noted that this formalism was capable of other interpretations. Already in the 1930s Kurt Gödel had observed that the box operator of modal logic can be read as "it is provable in the system S that", given that S is a suitable formal system. In the 1950s Georg Henrik von Wright championed several other interpretations: "epistemic", "doxastic", "deontic" (some ten years later the former two would be extensively explored by Jaakko Hintikka), and Arthur Prior developped "tense-logic" in close analogy with modal logic.

However, it was only with Saul Kripke that modal logic really took off. Beginning 1959 he published several papers in which he introduced what we now refer to as Kripke semantics or possible-worlds-semantics. Historians interested in the development of modal logic will have to assess the relative importance of Carnap's and Prior's work as well as the work of Stig Kanger and Jaakko Hintikka, who published related ideas independently of Kripke and in fact somewhat earlier than he; there is also the famous Jónson & Tarski paper from 1951. Nevertheless, there is no doubt that it was Kripke's papers that triggered the explosive growth of modal logic of the following two decades.

The exposition in the present notes is in the tradition of John Lemmon and Dana Scott as set out in the Lemmon Notes. One feature that makes their theory so elegant is the concept of the canonical model. The idea of using Henkin's method in modal logic occurred, independently, to a number of other authors as well, for example, David Makinson, Max Cresswell and Kurt Schütte, but those authors restricted themselves to case studies and did not see and did not seek the generality that Lemmon and Scott achieved. The concept of filtration, which they also employed, was modelled on an algebraic construction of J. C. C. McKinsey.

In the bibliography four textbooks have been listed. Lemmon's book, the published version of a draft completed three days before Lemmon's death, is of great historical interest. Written as a monograph rather than as a textbook perhaps it makes greater demands on readers that the other three, but it is still a favourite with this author. The books by Chellas and Hughes & Cresswell are standard texts in modal logic. Goldblatt's book, unlike the other three, deals with dynamic logic as well as modal logic. Therefore it is probably the best choice for those whose interest in modal logic is secondary to their interest in dynamic logic.

Lemmon's book contains a valuable historical introduction. Some historical remarks are also made in the survey article by Robert Bull and the author.

In section 2.2.2 we touched on tense logic. For further discussion, see the survey paper by Burgess listed in the bibliography. The author's paper "On von Wright's tense-logic", also listed in the bibliography, was to have been the first publication of a completeness proof for the tense-logic of discrete linear future time with operators for both 'next' and 'at all times'.

2. Dynamic logic. There are shorter completeness proofs for PDL than the one given here. The virtue of our proof is that it so clearly belongs in the tradition of modal logic: from a theoretical point of view, dynamic logic is a generalization of modal logic. Consequently the techniques that modal logicians have built up are almost immediately available for studying dynamic logic.

At the very beginning, this connexion with modal logic was not obvious to the founder of the discipline, Vaughan Pratt. In the mid-70s Pratt, then assistant professor of computer science at MIT, was teaching a course in which program verification was one issue. There is a long tradition of computer scientists who have tried to develop useful formalisms for reasoning about what programs do. Pratt, trying to improve on previous efforts, developped his own theory, which prompted one well-read student to come up after one class and suggest that what Pratt was doing was just modal logic. Incredulously Pratt checked out Hughes & Cresswell's *Introduction to modal logic* from the library. After spending a week-end with that classic, Pratt was convinced: there was a clear connexion with modal logic.

Even after the basic semantics was worked out, it was not clear how to develop the theory. The first result in dynamic logic (at that time still called the modal logic of programs) seems to have been due to Michael Fischer and Richard Ladner, who were able to prove in 1976 that PDL—the set of formulæ valid in all standard frames—has the strong fmp and so is decidable.

In modal logic it is unsual for an fmp result to be proved before completeness has been settled, but in this case completeness turned out to be hard. By the summer of 1977 the author of these notes had worked out the completeness of ancestral logic (essentially the proof presented in chapter 2). He had also developped a completeness proof for PDL, which he presented in Brian Chellas's seminar at the University of Calgary in July 1977 and then announced in the *Notices of the A. M. S.* Independently of this and of one another, several other researchers were trying to produce their own completeness proofs. In particular, Rohit Parikh, then at Boston University, had his own proof by November 1977. In early January 1978 the author, to his everlasting chagrin, discovered that one of his inductions did not get off the ground. In other words, his proof contained a gap and therefore was no proof. The author's completeness proof for ancestral logic was still correct, but the honour of having produced the first correct proof for dynamic logic belongs to Parikh. Later he and Dexter Kozen published a shorter proof, which is now regarded as the classic reference for the completeness of PDL. The author's mended proof, essentially the proof given here, was presented in March 1978 at the Banach Center in Warsaw.

The survey article by David Harel, an informative if difficult paper, provides an account of the intense period of work following the initial period described above.

4.2. Selective bibliography

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