An Essay in Classical Modal Logic

Krister Segerberg

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Chapter 1

Fundamental concepts

1.1 Basic syntax

This essay purports to deal with classical modal logic. The qualification "classical" has not yet been given an established meaning in connection with modal logic, and we shall explain in this section the sense in which the modal logics studied here are classical. Clearly one would like to reserve the label "classical" for a category of modal logics which – if possible – is large enough to contain all or most of the systems which for historical or theoretical reasons have come to be regarded as important, and which also possesses a high degree of naturalness and homogeneity. Such a category will be suggested. As it turns out, some well-known systems, like **S2** and **S3**, will not be classical under our definition. However, we also define a more inclusive category called "quasi-classical" which contains those and similar systems. Of systems that are not even quasi-classical in our sense we may mention **S0.5** and Prior's system **Q**. That those systems really are of a different character than the systems treated here seems arguable (which is not to say that they are not related to quasi-classical systems).

We will use the same object language throughout. The *primitive symbols* shall be the following (where $\mathbb{N}$ is the set of natural numbers, including 0):

i. For each $n \in \mathbb{N}$, the propositional letter p_n.
ii. The logical constants $\bot$ (zeroary) and $\to$ (binary).
iii. The modal operator $\Box$ (unary).

The set of propositional letters is denoted by $\mathsf{Var} = \{p_0, p_1, \ldots\}$. Our *formulas* are defined as follows:

i. Every propositional letter is a formula.
ii. $\bot$ is a formula.
iii. If A and B are formulas, then $\to AB$ is a formula
iv. If B is a formula, then $\Box A$ is a formula.
v. Nothing is a formula except as prescribed by (i)–(iv).

The set of all formulas is denoted by Fm. The *length* of a formula is its number of primitive symbols. Thus a formula is of length 1 iff it is either $\bot$ or a propositional letter.

For the sake of greater perspicuity we adopt the following conventions for abbreviating formulas:

$$\begin{aligned}
\top &:= \to \bot\bot \\
(A \to B) &:= \to AB \\
(\neg A) &:= (A \to \bot) \\
(A \wedge B) &:= \neg(A \to \neg B) \\
(A \vee B) &:= (\neg A \to B) \\
(A \leftrightarrow B) &:= ((A \to B) \wedge (B \to A)) \\
(\Diamond A) &:= (\neg(\Box(\neg A)))
\end{aligned}$$

The parentheses are not primitive symbols in this set-up: they occur on the meta-language only. We shall feel free to drop them or add them as clarity demands. Two general rules are always to omit outermost parentheses, and always regard $\neg$, $\Diamond$ as binding more strongly than $\wedge$, $\vee$, which in turn bing more strongly than $\to$, $\leftrightarrow$. For example, $\neg A \vee B \to C$ would abbreviate the same formula as $(((\neg A) \vee B) \to C)$.

We suggest the following locutions:

"truth" for $\top$	"A materially implies B" for $A \to B$
"falsity" for $\bot$	"A and B are materially equivalent" for $A \leftrightarrow B$
"not A" for $\neg A$	"necessarily A" for $\Box A$
"A and B" for $A \wedge B$	"possibly A" for $\Diamond A$
"A or B" for $A \vee B$	

Whereas these conventions are in accord with naive intuitions and usual practice, they are not part of our formal development; it would be possible to adopt quite different conventions, or none. In particular it must be emphasized that there are no metaphysical commitments implicit in reading "necessarily" for the "box operator" $\Box$ and "possibly" for the "diamond operator" D.

Our set of primitive symbols together with our set of formulas constitute our *language*. All logics discussed in this essay will use that language. It would be false to assert that all the important logics, such as Lewis's systems, have been formulated in it. However, practically all modal logics in the literature have obvious counterparts in our language. In this dissertation such a difference of primitive languages is not important (although [Cresswell, 1971] shows that it sometimes is). Thus we shall refer to, for example, **S4** as a logic having our primitive language, even though it was first defined by Lewis in a different primitive language.

1.1.1 Modal systems

By a *formal system* is usually understood an ordered triple $\langle \mathbb{L}, \mathbb{A}, \mathbb{R} \rangle$ such that

i. $\mathbb{L}$ is a language (that is, an ordered pair consisting of a nonempty set of primitive symbols and a nonempty set of formulas (that is, finite strings of primitive symbols));
ii. $\mathbb{A}$ is a set of formulas, called *axioms*;
iii. $\mathbb{R}$ is a set of rules, called (primitive) *inference rules*, which specify under what circumstances a formula may be inferred from a set of formulas.

All inference rules considered in this assay will be of the following general form:

$$\text{from } A_1, \ldots, A_n \text{ infer } B, \text{ provided that } \phi(A_1, \ldots, A_n, B),$$

where ϕ is a condition on the syntactic structure of the formulas $A_1, \ldots, A_n, B$. It is clear what a *derivation* or *proof* in a system S with such inference rules is going to be: any finite sequence $A_1, \ldots, A_k$ such that $k \geqslant 0$ and

i. A_1 is an axiom in S;
ii. whenever $1 \leqslant i \leqslant k$, either A_i is an axiom, or there is some rule R in S and some $j_1, \ldots, j_n$ such that $1 \leqslant j_1, \ldots, j_n < i$ and A_i can be inferred from $A_{j_1}, \ldots, A_{j_n}$ by virtue of R.

The *length* of a derivation is the number of formulas of which it consists. We say that a formula A is *derivable* in S or is a *theorem* of S iff there is a derivation $B_1, \ldots, B_k$ in S such that $B_k = A$. In symbols this is expressed by $\vdash_S A$. If two systems S, S' have the same language and every theorem of S is a theorem of S', then we say that S' is an *extension* of S; if some theorem of S' is not a theorem of S, the extension is *proper*. An inference rule R is a *derived rule* of a system if the addition of R to the set of primitive rules does not yield a proper extension.

A *modal system* is a formal system that satisfies the following requirements. First, its language is the language already specified. Second, its axioms will include each instance of the following types:

$$A \to (B \to A),$$
$$(C \to (A \to B)) \to ((C \to A) \to (C \to B)),$$
$$((A \to \bot) \to \bot) \to A.$$

Such instances are called *nonmodal axioms*; in addition there sill usually be others, called *modal axioms*. Third, MP (modus ponens) will be one inference rule:

$$\mathsf{MP} \qquad \text{From } A \text{ and } A \to B \text{ infer } B.$$

A modal system is *classical* if the rule RE (replacement of material equivalents) is also one inference rule:

$$\mathsf{RE} \qquad \text{From } A \leftrightarrow B \text{ infer } \Box A \leftrightarrow \Box B.$$

By **PC** we denote the smallest modal system; that is, the system whose axioms are the nonmodal axioms and whose only inference rule is MP. By **E** we denote the smallest classical system; that is, the system whose axioms are the nonmodal axioms and whose only inference rules are MP and RE. A modal system S is called *quasi-classical* iff every theorem of **E** is a theorem of S.

A formula is called a *tautology* if it is derivable in **PC**. It is clear that this is in accord with ordinary terminology (see [Church, 1956]).

Theorem 1.1.1 (Replacement theorem for classical logics)**.** *Suppose L is a classical system. Let B and B' be formulas such that $\vdash_L B \leftrightarrow B'$. Suppose A and A' are formulas such that A' is like A except for having an occurrence of B' in one place where A contains an occurrence of B. Then $\vdash_L A \leftrightarrow A'$.*

The proof, by induction on the length of A, is omitted.

Suppose A and B are formulas. By $A_n[B]$ we shall understand the formula obtained by replacing every occurrence of p_n in A by an occurrence of B. We shall say that $A_n[B]$ is obtained from A by *substitution* of B for p_n, and that $A_n[B]$ is a *substitution instance* of A. Similarly we define $A_{n_1,\ldots,n_k}[B_1,\ldots,B_k]$ as the formula which is obtained by replacing each occurrence of p_{n_i} in A by an occurrence of B_i, for all $1 \leqslant i \leqslant k$. We say that $A_{n_1,\ldots,n_k}[B_1,\ldots,B_k]$ is obtained from A by *simultaneous substitution* of B_1 for p_{n_1}, …, B_k for p_{n_k}.

We say that a schema is *derivable* in a system S if every substitution instance of the schema is derivable in the system. If Z and Z' are schemata, we say that Z *implies* Z' *in* S if every substitution instance of Z' is derivable in the system obtained by adding every substitution instance of Z as a new modal axiom to S. Two schemata are said to be *equivalent in* S if each implies the other in S, and *independent in* S if neither implies the other in S.

1.1.2 Modal logics

By a *modal logic* we shall understand any set L of formulas such that

i. every tautology is in L;
ii. L is closed under MP (that is, if $A, A \to B \in L$ then $B \in L$);
iii. L is closed under substitution (that is, if $A \in L$ then $A_n[B] \in L$ for all n and B).

Clearly, if S is a (classical) modal system for which substitution is a primitive or derived rule, then the set of theorem of S is a (classical) modal logic. Conversely, for every (classical) modal logic L there is some (classical) modal system whose set of theorems coincides with L (for example, trivially, the (classical) system whose axioms are the formula of L). Note that a modal logic L is classical iff L is closed under RE (that is, if $A \leftrightarrow B \in L$ then $\Box A \leftrightarrow \Box B \in L$). Similarly, a modal logic L is quasi-classical iff every theorem of **E** is a member of L.

In this essay we shall be more interested in what theorems a system has than in how a logic may be axiomatized. Therefore we shall often neglect the distinction between systems and logics. Strictly speaking, if S and S' are modal systems having the same set L of theorems but different sets of axioms or different sets of rules, then S and S' are distinct from one another (and, of course, from L). Notwithstanding, we shall usually identify S with S' and both with L. The reader should notice that this convention is not always adopted by other authors, and of course sometimes cannot be. For example, if one is mainly interested in comparing the relative strength of various axiom systems – as in the case in a good many papers on modal logic – then the distinction is vital.

Let L be a logic and Σ a set of formulas. We say that a formula A *follows from* Σ in L, and write $\Sigma \vdash_L A$, to mean that there are $B_1, \ldots, B_n \in \Sigma$, for some $n \geqslant 0$, such that $B_1 \ldots, B_n \to A$ is a theorem of L. (No parentheses are needed since $\mathbf{PC} \subseteq L$ and conjunction is known to be associative in **PC**.) Notice that this definition agrees with our previous convention for $\vdash$: with Σ the empty set $\varnothing$ we obtain $\varnothing \vdash_L A$ iff

$\vdash_L A$. Another way of expressing this definition is as follows: Let $L(Sigma)$ be the system whose axioms are $L \cup \Sigma$ and whose only inference rule is MP. Then $\Sigma \vdash_L A$ iff A is derivable in $L(\Sigma)$.

We note without proof the following fact:

Theorem 1.1.2 (Deduction theorem for modal logic)**.** *Suppose L is a modal logic, Σ a set of modal formulas, and A, B formulas. Then $\Sigma \cup \{A\} \vdash_L B$ if and only if $\Sigma \vdash_L A \to B$.*

1.1.3 Maximal consistent sets of formulas

A logic is (*absolutely*) *consistent* if $\bot$ is not a theorem of it, otherwise (*absolutely*) *inconsistent*. There exists only one absolutely inconsistent logic. A set of formulas Σ is said to be *L-inconsistent* iff $\Sigma \vdash_L \bot$, and *L-consistent* otherwise. Hence Σ is L-inconsistent iff every formula follows from Σ in L. A set of formulas Σ is *L-maximal* iff Σ is L-consistent and, for every formula A, $A \in \Sigma$ or $\neg A \in \Sigma$. Thus an L-consistent set Σ is L-maximal iff, for all A, if $A \notin \Sigma$ then $\Sigma \cup \{A\}$ is L-inconsistent. Another criterion of L-maximality is this: $L \subseteq \Sigma$ and, for every A, $A \in \Sigma$ iff $\neg A \notin \Sigma$. We observe that L-maximal sets Σ have the following properties:

$$\begin{array}{rll} L \subseteq \Sigma & & \\ \neg A \in \Sigma & \text{iff} & A \notin \Sigma \\ A \wedge B \in \Sigma & \text{iff} & A \in \Sigma \text{ and } B \in \Sigma \\ A \vee B \in \Sigma & \text{iff} & A \in \Sigma \text{ or } B \in \Sigma \\ A \to B \in \Sigma & \text{iff} & \text{if } A \in \Sigma \text{ then } B \in \Sigma \\ A \leftrightarrow B \in \Sigma & \text{iff} & A, B \in \Sigma \text{ or } A, B \notin \Sigma \end{array}$$

We recall the following basic fact:

Lemma 1.1.3 (Lindenbaum's Lemma)**.** *Let L be a consistent modal logic. Then every L-consistent set of formulas is included in an L-maximal set.*

If Σ and Σ' are sets of formulas and $\Sigma \subseteq \Sigma'$, then we say that Σ' is an *extension* of Σ. Thus, Lindenbaum's Lemma states that every L-consistent set has an L-maximal extension. The proof of this contention is well known.

Corollary 1.1.4. *$\Sigma \vdash_L A$ if and only if every L-maximal extension of Σ contains A.*

Proof. The "difficult" part follows from 1.1.2 and 1.1.3. $\square$

1.1.4 Regular and normal logics

Among the classical logics there are several subfamilies. We conclude this section by defining regular logics and normal logics.

A modal system is *regular* if

i. every instance of the schema K is a modal axiom:

   K $\quad \Box A \wedge \Box B \to \Box(A \wedge B)$;

ii. the rule RR (regularity) is an inference rule:

   RR $\quad$ From $A \to B$ infer $\Box A \to \Box B$.

A modal system is *normal* if (i) and (ii) are true and also

iii. the rule RN (necessity) is an inference rule:

   RN $\quad$ From A infer $\Box A$.

The smallest regular logic is denoted by **C**, the smallest normal logic by **K**. It is not difficult to see that **K** is an extension of **C**, which is an extension of **E**:

$$\mathbf{E} \subseteq \mathbf{C} \subseteq \mathbf{K}.$$

That the extensions are proper will be seen below.

It should perhaps be remarked that our sense of "normal" differs from that proposed by [Kripke, 1963] and which has been adopted by some authors. Kripke normality is somewhat stronger condition than normality in our sense, which would be termed "semi-normality" by followers of Kripke's terminology. It will be apparent from the semantics to be defined why the new concept of normality is preferable at least in this essay.

1.2 Basic (neighborhood) semantics

In Section 1.1 we introduced the concept of a classical logic. Our next concern will be to define a semantics adequate for this kind of logics.

By a (*neighborhood*) *frame* we understand an ordered pair $\langle W, \mathcal{N}\rangle$ such that

i. W is a set;
ii. $\mathcal{N}$ is a function $W \to \wp(\wp(W))$.

(Here $\wp(W)$ denotes the power set of W.) W is called the *domain* or *universe* of the frame, and the elements of W are called *points* or – rarely in this dissertation – *possible worlds*. If $w \in W$, we may use either $\mathcal{N}_w$ or $\mathcal{N}(w)$ to denote the value of $\mathcal{N}$ for the argument w. Thus, for each $w \in W$, $\mathcal{N}_w$ will be a set of subsets of W, called *the set of neighborhood s* of w.

If $F = \langle W, \mathcal{N}\rangle$ is a frame, then we call $\langle W, \mathcal{N}, V\rangle$ a *model* on F, and V a *valuation* in F, if V is a function $\mathsf{Var} \to \wp(W)$.

Suppose $M = \langle W, \mathcal{N}, V\rangle$ is a given model. The concept of *truth* in M of a formula A at a point w – in symbols, $M, w \models A$ — is defined as follows:

i. For each $n \in \mathbb{N}$, $\quad M, w \models p_n$ iff $w \in V(p_n)$.
ii. $M, w \not\models \bot$.
iii. $M, w \models A \to B$ iff if $M, w \models A$ then $M, w \models B$.
iv. $M, w \models \Box A$ iff there exists some $a \in \mathcal{N}_w$ such that

$$a \;=\; \{w \in W \mid M, w \models A\}.$$

For simplicity we shall often use the convention

$$\|A\|^M \;:=\; \{w \in W \mid M, w \models A\}.$$

Thus clause (iv) is also expressed by

$M, w \models \Box A$ iff there is some $a \in \mathcal{N}_w$ such that $a = \|A\|^M$,

or even shorter by

$M, w \models \Box A$ iff $\|A\|^M \in \mathcal{N}_w$.

If, and only if, a formula is not true at a point, it is said to be *false* at the point. Symbolically, $M, w \not\models A$ will express that A is false at w in M. (The symbol M is omitted when this can be done safely.) If a formula is true at every point of a model, it is said to be *true* in the model; otherwise it is *falsified* or *rejected* by the model. The expressions "holds" and "fails" are used synonymously with "is true" and "is false". A formula is *valid* in a frame if it is true in every model on the frame. It is valid in a class of frames if it is valid in each frame. A model M is a *model for* a set Σ of formulas if every formula of Σ is true in M. M is a *countermodel* for a formula A if A is false somewhere in M. A frame F is a *frame for* Σ if every formula in $\Sigma$is valid in F. Two models are *equivalent modulo* Σ if every formula in Σ is either true in both or rejected by both models. Two models are *equivalent* if they are equivalent modulo the set of all formulas.

Let L be a logic and $\mathcal{C}$ a class of frames. We say that L is *consistent with respect to* $\mathcal{C}$ if every frame in $\mathcal{C}$ is a frame for L; and that L is *complete with respect to* $\mathcal{C}$ if every formula valid in $\mathcal{C}$ is a theorem of L. If L is consistent and complete with respect to $\mathcal{C}$, then we say that L is *determined* by $\mathcal{C}$. If $\mathcal{C}$ consists of just one frame F, and $\mathcal{C}$ determines L, we shall say that F determines L. If L is determined by a class $\mathcal{C}$ of frames, and $\mathcal{C}$ is the class of frames satisfying a certain condition γ, then it may be said that L is determined by γ.

Theorem 1.2.1. *Suppose that L is a classical system. Let $\mathcal{C}$ be any class o frames. If every modal axiom of L is valid in $\mathcal{C}$, then L is consistent with respect to $\mathcal{C}$.*

Proof. The proof goes by induction on the length of derivations in L. Every nonmodal axiom is easily seen to be valid in $\mathcal{C}$. The modal axioms are valid in $\mathcal{C}$ by hypothesis.

Suppose A and $A \to B$ are valid in $\mathcal{C}$. Let M be any model on any frame in $\mathcal{C}$. Take any w in M. Then $M, w \models A$ and $M, w \models A \to B$. So, by truth definition, $M, w \models B$. Hence MP preserves validity in $\mathcal{C}$.

Suppose finally that $A \leftrightarrow B$ is valid in $\mathcal{C}$. Let M be any model on any frame in $\mathcal{C}$. Since $A \leftrightarrow B$ is true in M, $\|A\|^M = \|B\|^M$. Then $\Box A \leftrightarrow \Box B$ must hold at every point in M. Hence RE preserves validity in $\mathcal{C}$. $\Box$

1.2.1 Neighborhood canonical model

A proof that a logic is determined by a certain class of frames is usually called a completeness proof in the literature, and the result it establishes is called a completeness theorem. Many of the results proved in this essay are completeness theorems. All of them are proved by considering so-called canonical models. We shall now define that concept.

Let L be a classical logic. W_L shall then denote the set of all L-maximal sets of formulas. We agree to let $|A|_L$ denote the set of L-maximal sets of which A is a member (the subscript L will be dropped when this can be done safely.) Thus $W_L = |\top|$. For each $w \in W_L$ define

$$\mathcal{N}_L(w) \ := \ \{\, a \subseteq W_L \mid \text{for some formula } A,\ \ \Box A \in w \text{ and } a = |A| \,\}.$$

Finally, for each $n \in \mathbb{N}$, define $V_L(p_n) := |p_n|$.

By the (*neighborhood*) *canonical model* (for L), designated by $\mathfrak{M}_L$, we understand the triple $\langle W_L, \mathcal{N}_L, V_L \rangle$. That this is a model is readily seen.

Theorem 1.2.2 (Fundamental theorem for classical logics)**.**
Suppose L is a classical logic. Then, for all formulas A and all points w in $\mathfrak{M}_L$,

$$\mathfrak{M}_L, w \models A \quad \textit{if and only if} \quad A \in w.$$

Proof. The proof goes by induction on the length of A. If A is a propositional letter or $\bot$ or an implication, then there is no difficulty. Suppose A is of the form $\Box B$ and that the theorem holds for B and all points v in $\mathfrak{M}_L$.

First, assume that $\Box B \in w$. Then, by definition of $\mathcal{N}_L$, $|B| \in \mathcal{N}_L(w)$. According to the induction hypothesis, $|B| = \|B\|$. Hence $\|B\| \in \mathcal{N}_L(w)$, so $\mathfrak{M}_L, w \models \Box B$.

Conversely, assume that $\mathfrak{M}_L, w \models \Box B$. Then, by the truth definition, there is some $a \in \mathcal{N}_L(w)$ such that $a = \|B\|$. Hence, by induction hypothesis, $a = |B|$. Since $a \in \mathcal{N}_L(w)$, there must exist some formula C such that $\Box C \in w$ and $a = |C|$. Thus, $|B| = |C|$. In other words, the formula $B \leftrightarrow C$ is contained in every L-maximal set of formulas. Hence, by Corollary 1.1.4, $\vdash_L B \leftrightarrow C$. Since L is classical, L is closed under the rule RE, so $\vdash_L \Box B \leftrightarrow \Box C$. Since $\Box C \in w$, it follows that $\Box B \in w$. $\Box$

We are now ready for our first completeness theorem, settling the completeness question for **E**, the smallest classical logic.

Theorem 1.2.3 (Completeness of **E**)**.** *The logic* **E** *is determined by the class of all frames.*

Proof. By Theorem 1.2.1, **E** is consistent with respect to the class of all frames. For completeness, suppose that A is a formula valid in every frame. Then in particular A must be true in the canonical model $\mathfrak{M}_{\mathbf{E}}$ for **E**. Hence, by the Fundamental Theorem and Corollary 1.1.4, $\vdash_{\mathbf{E}} A$. $\Box$

1.2.2 Semantics for regular and normal logics

The remainder of this section is devoted to the problem of finding semantics adequate for regular and normal logics. Suppose $F = \langle W, \mathcal{N} \rangle$ is a frame. We shall say that a point $w \in W$ is *normal* (in F) if $\mathcal{N}_w$ is a filter; that is, $\mathcal{N}_w \neq \varnothing$ and, for all $a, b \subseteq W$,

$$a, b \in \mathcal{N}_w \quad \text{iff} \quad a \cap b \in \mathcal{N}_w.$$

If $\mathcal{N}_w = \varnothing$, we shall say that w is *singular* (in F). A point that is either normal or singular is called *regular* (in F). A frame F itself is called *normal*, *singular*, or *regular* according to whether the points are all normal, all singular, or all regular. Thus normal frames are regular, as are singular frames.

Theorem 1.2.4. *Suppose L is a regular (normal) system, and let $\mathcal{C}$ be any class of regular (normal) frames. If every modal axiom of L is valid in $\mathcal{C}$, then L is consistent with respect to $\mathcal{C}$.*

Proof. In view of the proof of Theorem 1.2.1, it is sufficient to check that every instance of K is valid in every regular frame, and that the rules RR and RN preserve validity in regular and normal frames, respectively. To do that is straightforward. □

Suppose $F = \langle W, \mathcal{N} \rangle$ is a frame. By the *augmentation* of F we understand the frame $F^+ = \langle W, \mathcal{N}^+ \rangle$ such that, for all $w \in W$,

i. $\mathcal{N}_w \subseteq \mathcal{N}_w^+$;
ii. if w is normal in F then $\mathcal{N}_w^+$ is closed under supersets
(that is, if $a, b \subseteq W$ and $a \subseteq b$, then $a \in \mathcal{N}_w^+$ implies $b \in \mathcal{N}_w^+$);
iii. if w is normal in F, then $\bigcap \mathcal{N}_w^+ \in \mathcal{N}_w^+$;
iv. if w is singular in F then w is singular in F^+.

In other words, for all $w \in W$,

$$\mathcal{N}_w^+ = \begin{cases} \varnothing, & \text{if } \mathcal{N}_w = \varnothing; \\ \{a \subseteq W \mid \bigcap \mathcal{N}_w \subseteq a\}, & \text{if } \mathcal{N}_w \neq \varnothing. \end{cases}$$

A frame identical with its own augmentation is said to be *augmented.* Note that if a frame is augmented then it is regular. The concepts of augmentation and augmented apply to models in the obvious way.

Theorem 1.2.5. *Suppose L is a regular logic. Let $\mathfrak{M}_L$ be the canonical model for L and $\mathfrak{M}_L^+$ the augmentation of $\mathfrak{M}_L$. Then, for all A and all $w \in W_L$,*

$$\mathfrak{M}_L, w \models A \quad \textit{if and only if} \quad \mathfrak{M}_L^+, w \models A.$$

Proof. Assume that w is normal; then $\bigcap \mathcal{N}_w$ exists. To prove the theorem it will be enough to show that if B is a formula such that $\bigcap \mathcal{N}_w \subseteq |B|$, then $\Box B \in w$. Make the assumption. It is clear that

$$\bigcap \mathcal{N}_w \ = \bigcap \{\, |A| \colon \Box A \in w \,\}.$$

Therefore, by Corollary 1.1.4, $\{A \mid \Box A \in w\} \vdash_L B$. Hence there are some $A_1, \ldots, A_n$ such that $\Box A_1, \ldots \Box A_n \in w$ and $\vdash_L A_1 \wedge \ldots \wedge A_n \to B$. Since u is normal, we may assume that $n \geqslant 1$. The logic L is regular, so we can apply the rule RR and conclude $\vdash_L \Box A_1 \wedge \ldots \wedge \Box A_n \to \Box B$.

Repeated application of the scheme K, every instance of which is derivable in L, show that $\Box(A_1 \wedge \ldots \wedge A_n) \in w$. Hence also $\Box B \in w$, and the proof is complete. □

A completeness result for **C**, the smallest regular logic, follows easily from Theorems 1.2.4 and 1.2.5:

Theorem 1.2.6 (Completeness of **C**)**.** *The logic* **C** *is determined by the class of all regular frames.*

A completeness result for **K**, the smallest normal logic, also easily follows. Consistency with respect to the class of all normal frames is clear from Theorem 1.2.4. For completeness, suppose a formula is not derivable in **K**: then it fails somewhere in the canonical model $\mathfrak{M}_L$, and hence somewhere in the augmentation $\mathfrak{M}_L^+$ of $\mathfrak{M}_L$. Note that $\Box\top$ is a theorem of **K**. Hence, for every $w \in W_{\mathbf{K}}$, $W_{\mathbf{K}} \in \mathcal{N}_{\mathbf{K}}(w)$. Thus no point in $\mathfrak{M}_{\mathbf{K}}$ is singular, and therefore no point in $\mathfrak{M}_{\mathbf{K}}^+$ is. Consequently, since $\mathfrak{M}_{\mathbf{K}}^+$ is regular, $\mathfrak{M}_{\mathbf{K}}^+$ is normal. Thus we have proved:

Theorem 1.2.7 (Completeness of **K**)**.** *The logic* **K** *is determined by the class of all normal frames.*

Using Theorem 1.2.6 it is easy to prove that **C** is a proper extension of **E**. Specifically, it is easy to find a model that rejects some instance of the schema K; thus **E** is not regular. Similarly, Theorem 1.2.7 may be used to show that **C** is not normal, so that **K** is a proper extension of **C**.

1.2.3 Relationship to Kripke semantics

The reader familiar with Kripke-type semantics will wonder what the connection is between that and the neighborhood semantics we have been studying in this section. We shall have a few comments to offer.

Suppose $F = \langle W, \mathcal{N} \rangle$ is a regular frame. Then by *the alternative relation induced by F* we mean the binary relation R on W such that, for all $w, x \in W$,

$$w\,R\,x \quad \text{iff} \quad w \text{ is normal and } x \in \bigcap \mathcal{N}_w.$$

Theorem 1.2.8. *Let $F = \langle W, \mathcal{N} \rangle$ be a regular frame and R the alternative relation induced by F. Then, for all $w, x \in W$ such that w is normal, the following conditions are equivalent:*

i. *$w\,R\,x$;*
ii. *for all models M on F, for all formulas A, if $M, w \models \Box A$ then $M, x \models A$.*

Proof. To see that (i) implies (ii), assume that $w\,R\,x$ and $M, w \models \Box A$, for some model M on F and some formula A. Then $\|A\| \in \mathcal{N}_w$, hence $\bigcap \mathcal{N}_w \subseteq \|A\|$, hence $x \in \|A\|$.

To see that (ii) implies (i), assume that $w\,R\,x$ does not obtain. As w is normal, $\bigcap \mathcal{N}_w$ exists. hence $x \notin \bigcap \mathcal{N}_w$, so $x \notin a$ for some $a \in \mathcal{N}_w$. Take any valuation V in F such that $V(p_0) = a$. Let M be the model on F defined by V. Then $M, w \models \Box p_0$ and $M, x \not\models p_0$. $\Box$

Theorem 1.2.8 suggests an alternative semantics. Let $F = \langle W, R, Q \rangle$ be called a *relational frame* if $R \subseteq W \times W$ and $Q \subseteq W$. R is called the *alternative relation* of F; the set $\{x \mid w\,R\,x\}$ is the set of *alternatives* of w. Q is the set of *singular* elements of F; the elements of $W \setminus Q$ are called *normal.* A relational frame is *normal* if all its elements are normal, *singular* if all its elements are singular.

Truth at a point w in a relational *model* M on a frame $F = \langle W, R, Q \rangle$ is defined by making just one change in the corresponding definition for neighborhood models: replace clause (iv) by

(iv') $M, w \models \Box A$ iff $w \notin Q$ and for all x, if $w\,R\,x$ then $M, x \models A$.

All other semantical concepts originally defined for neighborhood frames or models are automatically taken over by the relational frames and models. When confusion is not possible, we will use the terms "frame" and "models" with no qualification; the context will make it plain what is intended.

Suppose $= \langle W, \mathcal{N} \rangle$ is a regular neighborhood frame. Then $F^* = \langle W, R, Q \rangle$ is *the relational counterpart* of F if

i. F^* is a relational frame;
ii. R is the alternative relation induced by F;
iii. $Q = \{w \in W \mid w \text{ is a singular element in } F\}$.

The following result is a corollary of Theorem 1.2.8:

Theorem 1.2.9. *Let F be a regular neighborhood frame and F^* its relational counterpart. Then F and F^* are equivalent if F is augmented.*

Corollary 1.2.10. *The logic* **C** *is determined by the class of relational frames.*

Corollary 1.2.11. *The logic* **K** *is determined by the class of normal relational frames.*

It is easy enough to see how to construct for any relational frame an augmented neighborhood frame whose relational counterpart it is. Hence, by Theorem 1.2.9, neighborhood semantics is at least as strong as relational semantics: if a logic is determined by a class of relational frames, there is a class of neighborhood frames that also determines it. That no class of relational frames determines a logic that is not regular is clear, so neighborhood semantics is in fact stronger than relational semantics. Yet it is possible that the two types of semantics are equally strong with respect to regular logics; this is a problem which is not solved in this essay.

Suppose L is a regular logic. Let $\mathfrak{M}_L = \langle W_L, R_L, Q_L, V_L \rangle$ be the relational counterpart of the neighborhood canonical model for L. We shall call this $\mathfrak{M}_L$ the (*relational*) *canonical model* for L. Note that W_L and V_L have the same meaning as before, and that

$$\begin{array}{rcl} w\, R_L\, x & \text{iff} & w \notin Q_L \text{ and for all } A, \text{ if } \Box A \in w \text{ then } A \in x; \\ w \in Q_L & \text{iff} & \Box\top \notin w. \end{array}$$

Theorem 1.2.12 (Fundamental theorem for regular logics)**.** *Suppose L is a regular logic. Let $\mathfrak{M}_L = \langle W_L, R_L, Q_L, V_L\rangle$ be the relational canonical model for L. Then, for all formulas A and all points $w \in W_L$,*

$$\mathfrak{M}_L, w \models A \quad \textit{if and only if} \quad A \in w.$$

When we deal with normal logics later, all frames and models will be relational and normal. It would then be a burden always to have to put $\varnothing$ for Q in the structures, and we will simply drop Q in those contexts. Thus we will identify $\langle W, R, \varnothing\rangle$ with $\langle W, R\rangle$, and $\langle W, R, \varnothing, V\rangle$ with $\langle W, R, V\rangle$. In particular, if L is a normal logic, we will usually write $\langle W_L, R_L, V_L\rangle$ for the relational canonical model for L.

Theorem 1.2.13 (Fundamental theorem for normal logics)**.** *Suppose L is a normal logic. Let $\mathfrak{M}_L = \langle W_L, R_L, V_L\rangle$ be the relational canonical model for L. Then, for all formulas A and all points $w \in W_L$,*

$$\mathfrak{M}_L, w \models A \quad \textit{if and only if} \quad A \in w.$$

1.3 Some meta-theorems

Under this heading we shall bring together some definitions and observations which have at least this in common that they will be needed later. Where the observations are at least fairly evident, the proofs will be omitted. Throughout the section, unless otherwise stated, where "frame" and "model" are used ambiguously, both the neighborhood and the relational reading are allowed.

1.3.1 Intersection of logics

Theorem 1.3.1. *The intersection of any class of classical (regular; normal) logics is itself a classical (regular; normal) logic.*

Theorem 1.3.2. *Every class of (regular; normal) neighborhood frames determines a unique classical (regular; normal) logic. Every class of relational frames determines a unique regular (normal) logic.*

If $\mathcal{C}$ is a class of frames, we sometimes let $\mathsf{Logic}(\mathcal{C})$ denote the logic determined by $\mathcal{C}$. If F is a frame, we may write $\mathsf{Logic}(F)$ for the logic determined by F.

Theorem 1.3.3. *IF I is an indexing class and, for each $i \in I$, L_i is a logic determined by a class $\mathcal{C}_i$ of frames, then the logic $\bigcap_{i\in I} L_i$ is determined by the class $\bigcup_{i\in I} \mathcal{C}_i$.*

In particular, if $\mathcal{C}$ and $\mathcal{C}'$ are classes of frames determining the logics L and L' respectively, then the logic $L \cap L'$ is determined by the class $\mathcal{C} \cup \mathcal{C}'$.

1.3.2 Reduction to a single frame

Usually the classes occurring in our completeness results are proper classes (for example, the class of all frames). This should not worry anybody, for we could dispense with proper classes of frames and only consider sets of frames. For suppose L is a logic determined by a class $\mathcal{C}$ of frames. Then, for each formula A that is not a theorem of L let F_A be a frame in $\mathcal{C}$ such that A is not valid in F_A. Clearly the collection of all F_A's is an at most denumerable set of frames that determines L. However, we can do even better. Suppose it is neighborhood frames we are considering. For each formula $A \notin L$, let $F_A = \langle W_A, \mathcal{N}_A\rangle$. Define a neighborhood frame $F = \langle W', \mathcal{N}'\rangle$, where

$$\begin{array}{lcll} W' & = & \{\, \langle A, w\rangle \mid w \in W_A \text{ and } A \notin L \,\}; & \\ \mathcal{N}'_{\langle A,w\rangle} & = & \{\, \{\langle A, x\rangle \mid x \in a\} \mid a \in \mathcal{N}_A(w) \,\}, & \quad \text{for each } \langle A, w\rangle \in W'. \end{array}$$

Then L is determined by F. Inspection of the construction of F yields this result:

Theorem 1.3.4. *Every logic determined by a class of classical (regular; normal) frames is determined by one single classical (regular; normal) frame.*

It may be noted that for each of the particular logics we have encountered so far, **E**, **C**, and **K**, each is determined by the frame of its canonical model. A logic possessing this property may be called *natural*. It is an interesting question whether there are unnatural logics; none has yet been discovered. A somewhat related question is whether there are logics that are not determined by any class of frames. The latter is probably the outstanding question in this area of modal logic at the present time.

1.3.3 Distinguishable model

A concept which will be important later is the following. A model M will be called *distinguishable* if and only if, for all points w, w' in M, if $w \neq w'$ then there is some formula A such that $M, w \models A$ and $M, w' \not\models A$. Evidently, canonical models are always distinguishable.

Theorem 1.3.5. *For every model there can be found an equivalent distinguishable model.*

Proof. We prove the theorem only for relational models. Let $M = \langle W, R, Q, V\rangle$ be one. We introduce the following binary relation on W:

$$x \sim y \qquad \text{iff for all } A,\ \ x \models A \text{ iff } y \models A.$$

It is clear that $\sim$ is a congruence relation with respect to R, Q, and, for each $n \in \mathbb{N}$, $V(p_n)$. Writing $w/{\sim}$ for the equivalence class under $\sim$ of an element $w \in W$, we may therefore define $M^\circ = \langle W^\circ, R^\circ, Q^\circ, V^\circ\rangle$, where:

$$\begin{array}{ccl} W^\circ & := & \{w/{\sim} \mid w \in W\}; \\ w/{\sim}\, R^\circ\, x/{\sim} & \text{iff} & \text{there are } w', x' \text{ such that } w \sim w',\ x \sim x', \text{ and } w'\, R\, x'; \\ Q^\circ & := & \{w/{\sim} \mid w \in Q\}; \\ V^\circ(p_n) & := & \{w/{\sim} \mid w \in V(p_n)\}, \quad \text{for all } n \in \mathbb{N}. \end{array}$$

It is clear that the model M° is distinguishable. That for all formulas A and all points $w \in W$,

$$M, w \models A \quad \text{if and only if} \quad M^\circ, w/{\sim} \models A$$

is proved by induction on the length of A. We give the inductive step for the case that $A = \Box B$, the contention being assumed to hold for B.

(1) Assume $M, w \not\models \Box B$. If $w \in Q$ then $w/{\sim} \in Q^\circ$ and hence $M^\circ, w/{\sim} \not\models \Box B$. If $w \notin Q$ then there exists some x such that $w\, R\, x$ and $M, x \not\models B$. By the inductive hypothesis, $M^\circ, x/{\sim} \not\models B$. Since $w\, R\, x$, it follows by the definition of R° that $w/{\sim}\, R^\circ\, x/{\sim}$. Hence $M^\circ, w/{\sim} \not\models \Box B$.

(2) Assume $M^\circ, w/{\sim} \not\models \Box B$. If $w/{\sim} \in Q^\circ$ then $w \in Q$ and hence $M, w \not\models \Box B$. If $w/{\sim} \notin Q^\circ$ then there exists some $x/{\sim}$ such that $w/{\sim}\, R^\circ\, x/{\sim}$ and $M^\circ, x/{\sim} \not\models B$. By the definition of R° it follows that there are w', x' such that $w \sim w'$, $x \sim x'$, and $w'\, R\, x'$. The inductive hypothesis then yields $M, x' \not\models B$ and hence $M, w' \not\models \Box B$. It then follows that $M, w \not\models \Box B$. □

Lemma 1.3.6. *If M is a finite distinguishable model, then for every w in M one can find a formula A such that, for all x in M,*

$$M, x \models A \quad \textit{if and only if} \quad x = w.$$

Proof. Let $x_1, \ldots, x_n$ be all the elements of M. For all i, j, if $x_i \neq x_j$ find a formula A_{ij} such that $M, x_i \models A_{ij}$ and $M, x_j \not\models A_{ij}$; this is possible because M is distinguishable. For each i, let $A_{ii} = \top$. Then $M, x_j \models A_{i1} \wedge \ldots \wedge A_{in}$ iff $i = j$. □

Theorem 1.3.7. *Suppose L is a logic and F a finite frame. Suppose M is a distinguishable model on F. Then F is a frame for L if and only if M is a model for L.*

Proof. We prove the theorem for relational F and M. If F is a frame for L, then any model on F is a model for L. To prove the converse, suppose $F = \langle W, R, Q\rangle$ is a finite frame that is not a frame for L. Then there exists some formula A that is not a theorem of L and that fails somewhere in some model $M = \langle W, R, Q, V\rangle$. Since by hypothesis W is finite, $V(p_n)$ is finite, for every $n \in \mathbb{N}$. Thus, for every $n \in \mathbb{N}$, we may write $V(p_n) = \{x^n_1, \ldots, x^n_{i_n}\}$, where $i_n \geqslant 0$. (If $i_n = 0$ then $V(p_n) = \varnothing$.)

Let $M' = \langle W, R, Q, V'\rangle$ be any distinguishable model on F. By the preceding lemma we may find, for each $w \in W$, a formula B_w that is true in M' at w, and at w only.

We recursively define a formula C^* for each formula C:

i. for each $n \in \mathbb{N}$, put $p_n^* := B_{x_1^n} \vee \ldots \vee B_{x_{i_n}^n}$;
ii. $\bot^* = \bot$;
iii. $(C \to D)^* = C^* \to D^*$;
iv. $(\Box C)^* = \Box C^*$.

(Here we assume the convention that $p_n^* = \bot$ if $i_n = 0$.) We see that C^* is simply the formula resulting from simultaneous substitution in C of p_n^* for p_n, for all n such that p_n occurs in C. We now claim that, for all $w \in W$ and all formulas C,

$$M, w \models C \quad \text{if and only if} \quad M', w \models C^*.$$

The proof is by induction on the length of C. For the basic step it suffices to note that, for every $n \in \mathbb{N}$,

$$\begin{aligned} M, w \models p_n \quad &\text{iff} \quad w \in V(p_n) \\ &\text{iff} \quad w \in \{x_1^n, \ldots, x_{i_n}^n\} \\ &\text{iff} \quad w = x_1^n \text{ or } \ldots \text{ or } w = x_{i_n}^n \\ &\text{iff} \quad M', w \models B_{x_1^n} \text{ or } \ldots \text{ or } M', w \models B_{x_{i_n}^n} \\ &\text{iff} \quad M', w \models B_{x_1^n} \vee \ldots \vee B_{x_{i_n}^n} \\ &\text{iff} \quad M', w \models p_n^*. \end{aligned}$$

The inductive step is trivial. Thus, since A is rejected by M, A^* is rejected by M'. Since A is derivable in L, A^* is also so derivable, for L is closed under substitution. Therefore M' is not a model for L, which proved the theorem. □

1.3.4 Finite model property

We shall say that a logic L has the *finite frame property* (f.f.p.) if, for every nontheorem A of L, there is a finite frame for L in which A is not valid. Similarly, L has the *finite model property* (f.m.p.) if every nontheorem of L is rejected by some finite model for L. The main interest of Theorem 1.3.7 lies in the fact that it shows that these notions are co-extensive:

Corollary 1.3.8. *A classical logic has the finite frame property if and only if it has the finite model property.*

The importance of these notions is that they enable us to form a simple criterion of decidability. A set Σ of formulas is *decidable* if there is a procedure that, for each formula A, allows one to decide, in a finite number of steps, whether A is a theorem of L. A logic is decidable if the set of its theorems is. Let us say that a logic is *axiomatizable* if there is a system S such that

i. there is no inference rule other than MP and RE;
ii. the modal axioms of S are the instances of a finite number of schemata;
iii. The theorems of S are exactly the theorems of L.

We say that S is an *axiomatization* of L. If S satisfies also the further requirement

i'. the only inference rule is MP;

then S is a *finite axiomatization* of L. In this case we say that L is *finitely axiomatizable.*

Theorem 1.3.9. *A logic is decidable if it is axiomatizable and has the finite frame property.*

Proof. Say L is a logic satisfying both conditions. Since a derivation in L is a finite string of formulas, which in turn are finite strings of symbols, it is possible to define an enumeration of all derivations in L: $D_0, D_1, \ldots, D_n, \ldots$ It is also possible to define an enumeration of all finite models: $M_0, M_1, \ldots, M_n, \ldots$

The following is a procedure that, for each formula A, after only a finite number of steps, yields the answer YES or NO to the question whether A is a theorem of L:

Begin. Go to **Step 0.**
Step n. If n is even, check whether $D_{n/2}$ is a derivation for A.
If it is go to **Stop 1**, else go to **Step** n+1.
If n is odd, check whether $M_{(n-1)/2}$ is a model for L.
If it is, and if A is false in it, go to **Stop 2**, else go to **Step** n+1.
Stop 1. Stop! The answer is YES.
Stop 2. Stop! The answer is NO.

Here it is of course important to note that for any finite model one can check in a finite number of steps whether it is a model for a certain axiomatizable logic. This is so because it is enough to check the modal axioms of any axiomatization of the logic, and they are the instances of only finitely many schemata. □

The term "finite model property" was evidently coined by R. Harrop, who did not use the term "model" in exactly the same sense as we do. For discussions of Harrops concept in connection with modal logic, see [Lemmon, 1966a] and [Makinson, 1969].

1.3.5 Generated submodel

We conclude this section by proving two important results which will be used frequently in the sequel. Although there are somewhat analogous theorems for neighborhood models, these theorems concern relational models only. Thus we assume, for the remainder of the section, that all models and frames considered are relational.

Suppose $M = \langle W, R, Q, V\rangle$ is a model. Let $w \in W$. By the model M^w *generated from* M *by* w we understand the model $\langle W^w, R^w, Q^w, V^w\rangle$, where

$$\begin{array}{rcl} W^w & = & \{x\colon w\, R^*\, x\}; \\ R^w & = & R \cap (W^w \cap W^w); \\ Q^W & = & Q \cap W^w; \\ V^w(p_n) & = & V(p_n) \cap W^w, \quad \text{for each } n \in \mathbb{N}. \end{array}$$

(By R^* is meant the reflexive-transitive closure of R.) We shall say that M is *point generated* if it is generated by some point w, i.e., if $M = M_w$.

Theorem 1.3.10 (Generation theorem)**.** *Suppose M^w is a model generated from M by some w. Then for all formulas A and all $x \in W^w$,*

$$M, x \models A \quad \textit{if and only if} \quad M^w, x \models A.$$

The proof, by induction, is omitted.

Let us say that a regular logic is *monolithic* if it is determined by a single point generated frame. We shall settle below the question whether every logic is monolithic (the answer is negative).

1.3.6 p-morphism

Let $M = \langle W, R, Q, V\rangle$ and $M' = \langle W', R', Q', V'\rangle$ be models. Suppose f is a function from W to W' satisfying the following conditions, for all $w, x \in W$,

i. f is onto;
ii. if $w\, R\, x$ then $fw\, R'\, fx$;
iii. if $fw\, R'\, fx$ then there exists some y such that $fx = fy$ and $w\, R\, y$;
iv. $w \in Q$ iff $fw \in Q'$.

Then f is said to be a *p-morphism* from M to M'. Furthermore, f is said to be *reliable* on p_n if

$$f[V(p_n)] \;=\; V'(p_n).$$

The concept of p-morphism is slightly more general than that of homomorphism. It derives its interest from the following fact:

Theorem 1.3.11 (p-morphism theorem)**.** *Suppose f is a p-morphism from $M = \langle W, R, Q, V\rangle$ to $M' = \langle W', R', Q', V'\rangle$. Then for all formulas A such that f is reliable on every propositional letter occurring in A, and for all $w \in W$,*

$$M, w \models A \quad \textit{if and only if} \quad M', fw \models A.$$

Hence M and M' are equivalent modulo the set consisting of the formulas on every propositional letter of which f is reliable. In particular, if f is reliable on every propositional letter, then M and M' are equivalent.

The simple proof – by induction on A – is omitted.

1.4 Some extensions of E

We have already met with two proper extensions of **E**: **C** and **K**. In this section we shall discuss these and several others, using the neighborhood semantics throughout. We shall use Lemmon's device for naming logics: whenever "X" names a schema or a formula and L is a name of a logic, then "LX" will name the logic arising if (every instance of) X is added as a new modal axiom to L; note that LX will have the same primitive inference rules as L. This is our basic convention; others will be introduced later.

First a remark on completeness proofs, which constitute the bulk of this essay. They always consist of two parts, the consistency part (sometimes also called the soundness part) and the completeness part proper. The consistency part consists in proving that all theorems of the logic under examination are valid in the class of frames with respect to which determination is to be proved. To do this for a classical logic it is enough to check whether the modal axioms are thus valid (remember Theorems 1.2.1 and 1.2.4). For the most part such checks are very simple to carry out, and we shall rarely reproduce them here. The completeness part, on the other hand, is usually a much more difficult affair. Roughly speaking, this one is also easy when it is easy to prove that the logic in question is natural, that is, when the canonical model is a frame for the logic; for then, by the Fundamental Theorem, completeness is immediate.

To illustrate these remarks we consider some logics definable in terms of the following formulas:

N. $\Box\top$
$\mathsf{Q_0}$. $\Box\bot$
D. $\Diamond\top$
$\mathsf{S_0}$. $\Diamond\bot$

For consistency we note that each formula is valid in any frame $\langle W, \mathcal{N}\rangle$ satisfying a particular condition:

for N the condition that $W \in \mathcal{N}_w$;
for $\mathsf{Q_0}$ the condition that $\varnothing \in \mathcal{N}_w$;
for D the condition that $\varnothing \notin \mathcal{N}_w$;
for $\mathsf{S_0}$ the condition that $W \notin \mathcal{N}_w$.

(The conditions are expressed by open statements. Thus, for example, $W \in \mathcal{N}_w$ has the same import as $\forall w\,(W \in \mathcal{N}_w)$.)

For completeness we note that, for each of the formulas, whenever it is derivable in a consistent classical logic L, then the canonical model $\mathfrak{M}_L$ satisfies the corresponding condition:

if **EN** $\subseteq L$ then $W_L \in \mathcal{N}_L(w)$;
if $\mathbf{EQ_0} \subseteq L$ then $\varnothing \in \mathcal{N}_L(w)$;
if **ED** $\subseteq L$ then $\varnothing \notin \mathcal{N}_L(w)$;
if $\mathbf{ES_0} \subseteq L$ then $W_L \notin \mathcal{N}_L(w)$.

Hence we immediately have four new completeness results, for **EN**, $\mathbf{EQ_0}$, **ED**, and $\mathbf{ES_0}$. In fact, the above remarks provide us with completeness results for all absolutely consistent proper extensions of **E** definable in terms of N, $\mathsf{Q_0}$, D, $\mathsf{S_0}$, namely (in addition to the four already mentioned) $\mathbf{ENQ_0}$, **END**, $\mathbf{EQ_0S_0}$, and $\mathbf{EDS_0}$.

An interesting trio of schemata is this:

T. $\Box A \to A$
Q. $\Box A$
S. $\Diamond A$

For consistency, note that each formula is valid in any frame $\langle W, \mathcal{N}\rangle$ satisfying a certain condition:

for T the condition that if $\mathcal{N}_w \neq \varnothing$ then $w \in \bigcap \mathcal{N}_w$;
for Q the condition that $\mathcal{N}_w = \wp(W)$;
for S the condition that $\mathcal{N}_w = \varnothing$.

It is easy to see that whenever L is a consistent classical extension of **ET** (resp. **ES**), then the canonical model $\mathfrak{M}_L$ satisfies the condition for T (resp. S). In the case of Q a weaker statement must be made: for every consistent classical extension L of **EQ** the augmented canonical model $\mathfrak{M}_L^+$ satisfies the condition for Q. From these remarks we easily derive completeness results for **ET**, **EQ**, and **ES**. **ETQ** and **EQS** are of course the inconsistent logic, and **EST** = **ES**. Notice that **ES** is the logic determined by the class of singular frames.

Completeness results of the kind we have now exemplified may not be very striking. However, via the filtration technique to be presented below we may use them to conclude that these logics are decidable and even have the finite model property. The completeness results also permit us to conclude that the logics studied are distinct. We shall exemplify the latter contention in connection with the next group of schemata to be studied.

1.4.1 The schema K and related schemata

Of the following schemata we have already encountered K; the others are new:

K. $\Box A \wedge \Box B \;\rightarrow\; \Box(A \wedge B)$
K'. $\Box(A \rightarrow B) \;\rightarrow\; (\Box A \rightarrow \Box B)$
R. $\Box(A \wedge B) \rightarrow \Box A$

It is easy to prove that the logic **EK** is determined by the condition

(k). if $a, b \in \mathcal{N}_w$ then $a \cap b \in \mathcal{N}_w$

and that the logic **ER** is determined by the condition

(r). if $a \cap b \in \mathcal{N}_w$ then $a \in \mathcal{N}_w$.

If L is a consistent classical extension of **EK** then the canonical model $\mathfrak{M}_L$ satisfies (k). If L is a consistent classical extension of **ER**, then the augmented canonical model $\mathfrak{M}_L^+$ satisfies (r). There is a corresponding completeness result for **EK**$'$, but, whereas the conditions (k) and (r) are familiar – see the definition of filter above – that for K' is unintuitive, and we abstain from giving it. Even without it we are able to discuss the relative strength of the three schemata.

Theorem 1.4.1. *The schema K' is not derivable in the logic* **EK**.

Proof. Let $M = \langle W, \mathcal{N}, V \rangle$ be this model:

$$\begin{aligned} W &= \{0,1,2,3\}; \\ \mathcal{N}_i &= \{\{0\}, \{0,1\}, \{0,2,3\}\}, \quad \text{for } i = 0,1,2,3; \\ V(p_n) &= \begin{cases} \{0,1\}, & \text{if } n = 0, \\ \{0,2\}, & \text{if } n > 0. \end{cases} \end{aligned}$$

Then $0 \models \Box p_0$, $0 \models \Box(p_0 \rightarrow p_1)$, and $0 \not\models \Box p_1$, which violates K'. Yet M is a model for **EK**, since each $\mathcal{N}_i$ is closed under intersection and thus satisfies (k). □

Theorem 1.4.2. *The schema K' is not derivable in the logic* **ER**.

Proof. Let $M = \langle W, \mathcal{N}, V \rangle$ be this model:

$$\begin{aligned} W &= \{0,1,2,3\}; \\ \mathcal{N}_i &= \{\{0,1\}, \{0,1,2\}, \{0,1,3\}, \{0,2,3\}, \{0,1,2,3\}\}, \quad \text{for } i = 0,1,2,3; \\ V(p_n) &= \begin{cases} \{0,1\}, & \text{if } n = 0, \\ \{0,2\}, & \text{if } n > 0. \end{cases} \end{aligned}$$

Then $0 \models \Box p_0$, $0 \models \Box(p_0 \rightarrow p_1)$, and $0 \not\models \Box p_1$, which violates K'. However, each $\mathcal{N}_i$ is closed under superset and hence satisfies (r) and therefore M is a model for **ER**. □

Theorem 1.4.3. *Neither the schema K nor R is derivable in the logic* **EK**$'$.

Proof. Let $M = \langle W, \mathcal{N}, V \rangle$ be this model:

$$\begin{array}{rcl} W & = & \{0,1,2,3\}; \\ \mathcal{N}_i & = & \{\{0,1\}, \{0,2\}\}, \quad \text{for } i = 0,1,2,3; \\ V(p_n) & = & \begin{cases} \{0,1\}, & \text{if } n = 0, \\ \{0,2\}, & \text{if } n > 0. \end{cases} \end{array}$$

Then $0 \models \Box p_0$, $0 \models \Box p_1$, and $0 \models \Box(p_0 \wedge p_1)$; this violates K. Moreover, $0 \models \Box((p_0 \vee p_1) \wedge p_1)$ and $0 \not\models \Box(p_0 \vee p_1)$; this violates R.

Finally, M is a model for **EK**$'$. For suppose A and B are formulas such that, for some $i \in \{0,1,2,3\}$, $i \models \Box(A \to B)$ and $i \models \Box A$. There are two cases to consider.
(1) $\|A\| = \{0,1\}$. Then $\{2,3\} \subseteq \|A \to B\|$. But then $\|A \to B\| \notin \mathcal{N}_i$, which is absurd.
(2) $\|A\| = \{0,2\}$. Then $\{1,3\} \subseteq \|A \to B\|$. Again $\|A \to B\| \notin \mathcal{N}_i$, which is absurd.
Consequently, every instance of K$'$ holds vacuously in M, which is therefore a model for **EK**$'$. □

That K and R are independent in **E** is easy to see, and we omit the simple proofs.

Our discussion indicates that, in a sense, K is a more fundamental schema than K$'$; yet it is K$'$ that is most often used in axiomatizations of normal logics.

Note that it was not necessary to introduce inference rules other than RE in order to define regular and normal logics. In fact:

Theorem 1.4.4. *A classical logic is regular if and only if the schemata* K *and* R *are derivable; normal if and only if the schemata* K, R, *and* N *are derivable.*

The theorem follows from the following two lemmata:

Lemma 1.4.5. *The rule* RR *is derivable in the logic* **ER**.

Proof. Let us use the notation TF for "by truth-functional reasoning". Then our argument may be represented in the following manner:

1.	$A \to B$	by hypothesis derivable in **ER**;
2.	$A \leftrightarrow B \wedge A$	from 1 by TF;
3.	$\Box A \leftrightarrow \Box(B \wedge A)$	from 2 by RE;
4.	$\Box(B \wedge A) \to \Box B$	instance of R;
5.	$\Box A \to \Box B$	from 3 and 4 by TF. □

Lemma 1.4.6. *The rule* RN *is derivable in the logic* **EN**.

Proof.

1.	A	by hypothesis derivable in **EN**;
2.	$\top \leftrightarrow A$	from 1 by TF;
3.	$\Box\top \leftrightarrow \Box A$	from 2 by RE;
4.	$\Box\top$	the schema N;
5.	$\Box A$	from 3 and 4 by TF. □

Note that, even though RN is derivable in **EN**, RR is not. (It is easy to see that $\Box(p_0 \wedge p_1) \to \Box p_0$ is not a theorem of **EN**, whereas $(p_0 \wedge p_1) \to p_0$ is.) However, RN is a stronger rule than RR in **EK**$'$, in the sense that if RN is a derivable rule in any classical extension of **EK**$'$, then RR is also a derivable rule in that system. It would be interesting to know whether **EK**$'$ is the smallest logic with this property.

1.5 Some extensions of K

We devote this section to a discussion of some logics which will play important roles later, particularly in Chapter II. In order to simplify the exposition we restrict it to normal logics. The semantics is correspondingly restricted to normal relational frames.

1.5.1 The schemata $\mathsf{D}, \mathsf{T}, \mathsf{4}, \mathsf{B}, \mathsf{5}, \mathsf{G}, \mathsf{Lem}$

We first consider these schemata, of which two were also considered in the preceding section:

D.	$\Box\bot \to \bot$	$\mathsf{G_0}$.	$\Diamond(A \wedge \Box B) \to \Box(A \vee \Diamond B)$
T.	$\Box A \to A$	G.	$\Diamond\Box A \to \Box\Diamond A$
4.	$\Box A \to \Box\Box A$	$\mathsf{Lem_0}$.	$\Box(A \wedge \Box A \to B) \vee \Box(B \wedge \Box B \to A)$
B.	$A \to \Box\Diamond A$	Lem.	$\Box(\Box A \to B) \vee \Box(\Box B \to A)$
5.	$\Diamond A \to \Box\Diamond A$		

(A word on the names of these schemata. D stands for "deontic"; T refers to Feys' and von Wright's logic which was called "**t**" by Feys; 4 refers to Lewis' logic **S4**; G is in honor of Geach and Lem in honor of Lemmon; B refers to Kripke's "Brouwersche" system; E stands for "euclidean" (now it is 5 – E.Z.). For more information about the history of these schemata, see [Dummett:Lemmon, 1959] and [Lemmon:Scott, 1966].)

We introduce some definitions. Let $\langle W, R\rangle$ be a frame. We assume that the quantifiers range over W.

R is *serial* if $\exists x\,(w\,R\,x)$.
R is *convergent* if $x \neq y$ implies $\exists z\,(x\,R\,z \;\&\; y\,R\,z)$.
R is *strongly convergent* if $\exists z\,(x\,R\,z \;\&\; y\,R\,z)$.
R is *piecewise convergent* if $w\,R\,x \;\&\; w\,R\,y \;\&\; x \neq y$ implies $\exists z\,(x\,R\,z \;\&\; y\,R\,z)$.
R is *piecewise strongly convergent* if $w\,R\,x \;\&\; w\,R\,y$ implies $\exists z\,(x\,R\,z \;\&\; y\,R\,z)$.
R is *connected* if $x \neq y$ implies $x\,R\,y$ or $y\,R\,x$.
R is *strongly connected* if $x\,R\,y$ or $y\,R\,x$.
R is *piecewise connected* if $w\,R\,x \;\&\; w\,R\,y \;\&\; x \neq y$ implies $x\,R\,y$ or $y\,R\,x$.
R is *piecewise strongly connected* if $w\,R\,x \;\&\; w\,R\,y$ implies $x\,R\,y$ or $y\,R\,x$.
R is *euclidean* if $w\,R\,x \;\&\; w\,R\,y$ implies $x\,R\,y$.

This result is well known:

Theorem 1.5.1. *Suppose L is a normal logic. Then the following is true of the canonical model $\mathfrak{M}_L = \langle W_L, R_L, V_L\rangle$ for L:*

i. *If* $\mathbf{KD} \subseteq L$ *then* R_L *is serial.*
ii. *If* $\mathbf{KT} \subseteq L$ *then* R_L *is reflexive.*
iii. *If* $\mathbf{K4} \subseteq L$ *then* R_L *is transitive.*
iv. *If* $\mathbf{G_0} \subseteq L$ *then* R_L *is piecewise convergent.*
v. *If* $\mathbf{KG} \subseteq L$ *then* R_L *is piecewise strongly convergent.*
vi. *If* $\mathbf{KLem_0} \subseteq L$ *then* R_L *is piecewise connected.*
vii. *If* $\mathbf{KLem} \subseteq L$ *then* R_L *is piecewise strongly connected.*
viii. *If* $\mathbf{K5} \subseteq L$ *then* R_L *is euclidean.*
ix. *If* $\mathbf{KB} \subseteq L$ *then* R_L *is symmetric.*

The proofs are not difficult (see [Lemmon:Scott, 1966]).

Finding good names for modal systems is one of the more recalcitrant problems in the field. In addition to our basic principles described at the beginning of Section 1.4 we shall adopt the following practice:

D is a name of **KD**		**K4.2** is a name of $\mathbf{K4G_0}$	
T is a name of **KT**		**D4.2** is a name of $\mathbf{KD4G_0}$	
D4 is a name of **KD4**		**S4.2** is a name of $\mathbf{KT4G_0}$	
S4 is a name of **KT4**		**K4.3** is a name of $\mathbf{K4Lem_0}$	
S5 is a name of **KT4B**		**D4.3** is a name of $\mathbf{KD4Lem_0}$	
D45 is a name of **KD45**		**S4.3** is a name of $\mathbf{KT4Lem_0}$	
B is a name of **KTB** (the Brouwer system)			

Some but not all of these conventions are in accord with the literature. The main aberrance is from Sobociński's **K**-systems: his **K4** refers to a totally different system (see for example [Sobociński, 1964a]).

At the risk of boring the reader we list the following simple but important consequences of Theorem 1.5.1:

Theorem 1.5.2. *Suppose L is a normal logic. Let $M = \langle W, R, V\rangle$ be any point generated submodel of the canonical model for L. Then:*

i. *If* $\mathbf{K4} \subseteq L$ *then* R *is transitive.*
ii. *If* $\mathbf{D4} \subseteq L$ *then* R *is serial and transitive.*
iii. *If* $\mathbf{S4} \subseteq L$ *then* R *is reflexive and transitive.*
iv. *If* $\mathbf{K4.2} \subseteq L$ *then* R *is transitive and convergent.*
v. *If* $\mathbf{D4.2} \subseteq L$ *then* R *is serial, transitive, and convergent.*
vi. *If* $\mathbf{S4.2} \subseteq L$ *then* R *is reflexive, transitive, and convergent.*
vii. *If* $\mathbf{K4.3} \subseteq L$ *then* R *is transitive and connected.*
viii. *If* $\mathbf{D4.3} \subseteq L$ *then* R *is serial, transitive, and connected.*
ix. *If* $\mathbf{S4.3} \subseteq L$ *then* R *is reflexive, transitive, and connected.*
x. *If* $\mathbf{K45} \subseteq L$ *then* R *is transitive and euclidean.*
xi. *If* $\mathbf{K4B} \subseteq L$ *then* R *is transitive and symmetric.*
xii. *If* $\mathbf{D45} \subseteq L$ *then* R *is serial, transitive, and euclidean.*
xiii. *If* $\mathbf{S5} \subseteq L$ *then* R *is universal.*

The reader will notice that $\mathsf{G_0}$ and G are equivalent in $\mathbf{D}$, and $\mathsf{Lem_0}$ and Lem in $\mathbf{T}$. Thus, in particular:

$$\begin{aligned} \mathbf{D4.2} &= \mathbf{D4G} \\ \mathbf{S4.2} &= \mathbf{S4G} \\ \mathbf{S4.3} &= \mathbf{S4Lem} \end{aligned}$$

Also,

$$\mathbf{S5} = \mathbf{D4B} = \mathbf{T45}.$$

1.5.2 The schemata Alt_n

An important family of schemata that is not in the literature is the following, where n varies over the positive integers:

$\mathsf{Alt_n}$. $\quad \Box A_1 \vee \Box(A_1 \to A_2) \vee \ldots \Box(A_1 \wedge \ldots \wedge A_n \to A_{n+1})$.

Lemma 1.5.3. *If L is a normal logic, then every element in the canonical model $\mathfrak{M}_L = \langle W_L, R_L, V_L\rangle$ for L has $\leqslant n$ alternatives.*

Proof. Suppose that there are elements $w, x_1, \ldots, x_{n+1} \in W_L$ such that for all i, j, if $1 \leqslant i, j \leqslant n+1$, then $w\, R_L\, x_i$ and if $i \neq j$ then $x_i \neq x_j$. For each pair i, j such that $i \neq j$ there exists some formula A_{ij} such that $A_i, j \notin x_i$ and $A_i, j \in x_j$.

Let $B_i = A_{i,1} \vee \ldots \vee A_{i,i-1} \vee A_{i,i+1} \vee \ldots \vee A_{i,n+1}$. Then $B_i \in x_j$ iff $i \neq j$. This contradicts the fact that, by the new scheme $\mathsf{Alt_n}$, $\Box B_1 \vee \Box(B_1 \to B_2) \vee \ldots \vee \Box(B_1 \wedge \ldots \wedge B_n \to B_{n+1}) \in w$. □

One easily derives the following corollary:

Theorem 1.5.4. *The following logics are determined by the following conditions on frames $\langle W, R\rangle$:*

i. $\mathbf{KAlt_n}$: *each point has $\leqslant n$ alternatives.*
ii. $\mathbf{DAlt_n}$: *R is serial, and each point has $\leqslant n$ alternatives.*
iii. $\mathbf{K4Alt_n}$: *R is transitive, and W has $\leqslant n+1$ elements.*
iv. $\mathbf{D4Alt_n}$: *R is serial transitive, and W has $\leqslant n+1$ elements.*
v. $\mathbf{K45Alt_n}$: *R is transitive euclidean, and W has $\leqslant n+1$ elements.*
vi. $\mathbf{D45Alt_n}$: *R is serial transitive euclidean, and W has $\leqslant n+1$ elements.*
vii. $\mathbf{S4Alt_n}$: *R is reflexive transitive, and W has $\leqslant n$ elements.*
viii. $\mathbf{S5Alt_n}$: *R is universal, and W has $\leqslant n$ elements.*

It is possible to improve the result in Theorem 1.5.4 somewhat: in the cases (ii) , (iv), (vi)–(viii) the $\leqslant$ sign can be replaced by $=$. One way of establishing this is via filtrations (see Section 1.7).

Theorem 1.5.5. *Every normal sublogic of* $\mathbf{S5}$ *is infinite.*

Proof. Let L be any normal logic. Assume that L is finite; then there is some finite frame F determining L. Let n be an integer such that every element of F has at most n alternatives. Then $\mathsf{Alt_n}$ is valid in F, so $\mathsf{Alt_n}$ is a derivable schema in L. However, $\mathsf{Alt_n}$ is obviously not derivable in **S5**. Therefore L cannot be a sublogic of **S5**. □

This result can be generalized a good deal. Cf. [Dugundji, 1940].

1.6 Propositional functions and modalities

Suppose L is a fixed classical logic. Let Σ be a set of formulas. We shall say that a set Θ of formulas is a base for Σ (in L) if for every $A \in \Sigma$ there exists some $A' \in \Theta$ such that $\vdash_L A \leftrightarrow A'$. We shall say that Σ is logically finite (in L) if Σ has a finite base in L.

1.6.1 Propositional functions

Let Π be a set of formulas closed under substitution. Then Π is a *propositional function of n variables* ($n \geqslant 0$) if there exists some formula A containing exactly n distinct propositional letters such that $\{A\}$ is a base for Π. Suppose $p_{i_1}, \ldots, p_{i_n}$ are the propositional letters in such an A in order of their leftmost occurrence in A (this informal mode of expression is easily made exact). Then by *the value of* Π *for the arguments* $B_1, \ldots, B_n$ we would of course understand

$$\Pi(B_1, \ldots, B_n) \ := \ A_{i_1,\ldots,i_n}[B_1, \ldots, B_n].$$

There is no unique base for propositional functions except when $n = 0$; still the last definition is unambiguous for every n. Two propositional functions in n variables Π and Π' are *equivalent* (in L) if, for all formulas $A_1, \ldots, A_n$,

$$\vdash_L \ \Pi(A_1, \ldots, A_n) \leftrightarrow \Pi'(A_1, \ldots, A_n).$$

Theorem 1.6.1. *A classical logic L has only finitely many nonequivalent propositional functions in n variables if and only if the set $\Omega = \Omega(p_0, \ldots, p_{n-1})$ of formulas containing no other propositional letters than $p_0, \ldots, p_{n-1}$ is logically finite.*

Proof. First, suppose there are only finitely many nonequivalent propositional functions in n variables. Suppose there are r equivalence classes and let $\Pi_1, \ldots, \Pi_r$ be one representative from each class. Define

$$\Delta \ = \ \{\, \Pi_k(p_{i_1}, \ldots, p_{i_n}) \ \mid \ 1 \leqslant k \leqslant r,\ 0 \leqslant i_1, \ldots, i_n < n \,\}.$$

Δ is finite. Take any $A \in \Omega$. We may assume without loss of generality that A contains exactly n distinct propositional letters. (If not, let $p_{j_1}, \ldots, p_{j_t}$ be the propositional letters among $p_0, \ldots, p_{n-1}$ that are not in A. Define

$$A^* \ = \ A \wedge (p_{j_1} \to p_{j_1}) \wedge \ldots \wedge (p_{j_t} \to p_{j_t}).$$

Then $\vdash_L A \leftrightarrow A^*$, so if we find $A' \in \Delta$ such that $\vdash_L A^* \leftrightarrow A'$, then also $\vdash_L A \leftrightarrow A'$.) Then $\{A\}$ is the base of a propositional function Π in n variables. It follows from our hypothesis that there exists some m, $1 \leqslant m \leqslant r$, such that Π_m is equivalent to Π. Let $p_{j_1}, \ldots, p_{j_n}$ be the propositional letters in A in order of appearance. Then $\vdash_L \Pi_m(p_{j_1}, \ldots, p_{j_n}) \leftrightarrow A$, since $A_{j_1,\ldots,j_n}[p_{j_1}, \ldots, p_{j_n}] = A$. Clearly, $\Pi_m(p_{j_1}, \ldots, p_{j_n}) \in \Delta$. Hence Δ is a finite base for Ω.

Conversely, suppose that Ω is logically finite. Then there is some finite set Ω_0 that is a base for Ω. There is not loss of generality if we assume that $\Omega_0 \subseteq \Omega$ (for it this is not already true, we can replace each formula A in $\Omega_0 \setminus \Omega$ by a formula $A' \in \Omega$ such that $\vdash_L A \leftrightarrow A'$). Hence there are at most finitely many bases for propositional functions in at most n variables. □

1.6.2 Modalities

A propositional function Π of one variable is called a *modality* if Π has a base $\{A\}$ that satisfies the following conditions:

i. $A = \circ_1 \dots \circ_k p_n$, for some $k, n \in \mathbb{N}$;
ii. $\circ_1, \dots, \circ_k \in \{\neg, \Box, \Diamond\}$;
iii. if $i > 1$ then $\circ_i \neq \neg$.

Notice that although n is not uniquely defined, the $\circ_j$'s are. (In a way, this definition would have been more pleasant had $\neg$ and $\Diamond$ also been primitive. However, there is nothing unclear about the definition.) If $\circ_1 \neq \neg$, the modality is *affirmative*, otherwise *negative*. We agree to write $\Box^m$ and $\Diamond^m$ for strings of m necessity operators and m possibility operators respectively ($m \geqslant 0$). Thus if $\{A\}$ is the base of some affirmative modality Π, then there are unique $m_1, \dots, m_q$, $n_1, \dots, n_q \in \mathbb{N}$ such that

$$A \;=\; \Box^{m_1}\Diamond^{n_1} \dots \Box^{m_q}\Diamond^{n_q} p_i$$

for some i, provided we require that there are no consecutive zeros in the sequence $m_1, n_1, \dots, m_q, n_q$. The $2q$-tuple $\langle m_1, n_1, \dots, m_q, n_q\rangle$ is called the *index* of the modality Π.

Theorem 1.6.2. *A classical logic L has only finitely many nonequivalent modalities if and only if the set $\{\circ_1 \dots \circ_k p_0 \mid k \geqslant 0$ and $\circ_i \in \{\Box, \Diamond\}\}$ is logically finite.*

The proof is like that of Theorem 1.6.1.

Theorem 1.6.3. *Suppose L is a classical logic with only finitely many nonequivalcnt modalities (respectively, propositional functions in n variables). Suppose Ψ is a logically finite set of formulas. Let Ψ' be the closure of Ψ under modalities (respectively, propositional functions of n variables). Then Ψ' is logically finite.*

It should be clear what closure conditions are intended: Ψ is closed under modalities if whenever Π is a modality and $A \in \Psi$ then $\Pi(A) \in \Psi$; and Ψ is closed under propositional functions of n variables if whenever Π is a propositional function of n variables and $A_1, \dots, A_n \in \Psi$, then $\Pi(A_1, \dots, A_n) \in \Psi$. The proof of the theorem is omitted.

It is well known that **S4** has only finitely many nonequivalent modalities, namely fourteen; this was first proved in [Parry, 1939]. Hence all classical extensions of **S4** have also only finitely many nonequivalent modalities. This is a fact that will be of importance later.

Many systems have an infinitude of nonequivalent modalities. Thus it was proved in [Sobociński, 1953] that **T** belongs in that category, and [Sugihara, 1962] extended that result to a number of systems intermediate in strength between **T** and **S4**. An improvement of Sobociński's result in a different direction is found in [Lemmon:Scott, 1966]. where it is shown that the Brouwer system has infinitely many nonequivalent modalities. Also this result can be improved:

Theorem 1.6.4. *The logic* $\mathbf{BAlt_3}$ *has infinitely many nonequivalent modalities.*

Proof. Let $\mathbb{Z}$ be the set of all integers. It is easy to see that $\langle \mathbb{Z}, R\rangle$ is a frame for $\mathbf{BAlt_3}$ if R is defined in the following way: zRw iff $|z - w| \leqslant 1$. For then R is symmetric, and every point has exactly three alternatives. Define, for each $n \in \mathbb{N}$, a valuation $V(p_n) = \{z \mid z \geqslant 0\}$. Clearly, for all nonnegative z, $z \models \Box^m p_0$ iff $z \geqslant m$. Thus, for all nonnegative z and all positive k, $z \not\models \Box^z p_0 \to \Box^{z+k} p_0$. It follows that $\{\Box p_0\}$, $\{\Box\Box p_0\}$, ... are bases of infinitely many nonequivalent modalities. $\Box$

We conjecture – without being able to offer a proof at the present time – that Theorem 1.6.4 gives a best limit in the sense that no proper normal extension of $\mathbf{BAlt_3}$ has infinitely many nonequivalent modalities.

1.6.3 Hintikka schema

We conclude this section by stating a result, proved in [Lemmon:Scott, 1966], which we shall need to refer to in a later chapter.

Suppose $\langle W, R\rangle$ is a frame. For $n \geqslant 0$, we agree to write R^n for the relation such that

$$x\,R^n\,y \text{ iff for some } x_0,\dots,x_n, \text{ we have } x_0\,R\,x_1,\ \dots,\ x_{n-1}\,R\,x_n, \text{ and } x = x_0 \text{ and } y = x_n.$$

Thus, in particular, R^1 is the sane as R, and R^0 is the identity relation. Below it is sometimes convenient to write $y \in R^n(x)$ instead of $x\,R^n\,y$.

Suppose Π is an affirmative modality with index $\langle m_1, n_1, \dots, m_q, n_q\rangle$, which means that it corresponds to the sequence $\Box^{m_1}\Diamond^{n_1}\dots\Box^{m_q}\Diamond^{n_q}$. Then by R^{Π}_n, where $n \geqslant 0$, we understand the following relation:

$$\begin{aligned} x\,R^{\Pi}_n\,z \quad \text{iff} \quad & \forall x_1 \in R^{m_1}(z) \quad \exists y_1 \in R^{n_1}(x_1)\ \dots \\ & \forall x_q \in R^{m_q}(y_{q-1})\ \exists y_q \in R^{n_q}(x_q)\!:\ \ x\,R^n\,y_q. \end{aligned}$$

We call a schema a *Hintikka schema* if it is of the following type, where the Π's are supposed to be affirmative modalities, $r, s \geqslant 1$, and the other parameters are nonnegative integers:

$$\begin{aligned} \Diamond^{i_1}\Box^{j_1}A_1 \wedge \dots \wedge \Diamond^{i_r}\Box^{j_r}A_r \quad \longrightarrow \quad & \Box^{k_1}\Diamond^{\ell_1}(\Pi^1_1 A_1 \wedge \dots \wedge \Pi^1_r A_r)\ \vee \dots \vee \\ & \Box^{k_s}\Diamond^{\ell_s}(\Pi^s_1 A_1 \wedge \dots \wedge \Pi^s_r A_r). \end{aligned}$$

By *the corresponding Hintikka condition* we mean this condition, where x is the only free variable:

$$\begin{aligned} \forall x_1 \dots \forall x_r\ \Big(\, & x\,R^{i_1}\,x_1\ \&\ \dots\ \&\ x\,R^{i_r}\,x_r\ \longrightarrow \\ & \forall y_1 \in R^{k_1}(x)\ \exists z_1 \in R^{\ell_1}(y_1)\ [x_1\,R^{\Pi^1_1}_{j_1}\,z_1\ \&\ \dots\ \&\ x_r\,R^{\Pi^1_r}_{j_r}\,z_1]\ \ \vee \dots \vee \\ & \forall y_s \in R^{k_s}(x)\ \exists z_s \in R^{\ell_s}(y_s)\ [x_1\,R^{\Pi^s_1}_{j_1}\,z_s\ \&\ \dots\ \&\ x_r\,R^{\Pi^s_r}_{j_r}\,z_s]\Big). \end{aligned}$$

Theorem 1.6.5. *Let* H *be a Hintikka schema and* $h(x)$ *the corresponding Hintikka condition. Then the logic* $\mathbf{KH}$ *in determined by the condition* $\forall x\, h(x)$*.*

Let H^* *be the schema* $\Diamond\top \to \mathsf{H}$*. Then the logic* $\mathbf{KH}^*$ *is determined by the condition* $\forall x\,(\exists y\, xRy \to h(x))$*.*

This is an extremely general theorem (even though it is a special case of a more general theorem!): it covers most of the "ordinary" systems in the literature.

There is no point in reproducing the cumbersome proof of the theorem.

1.7 Filtrations

It is obvious what a *subformula* is. If a formal definition is desired, this one will do:

i. Every formula is a subformula of itself,
ii. Every subformula of A is a subformula of $A \to B$ and of $B \to A$.
iii. Every subformula of A is a subformula of $\Box A$.

A set Ψ of formulas is said to be *closed under subformulas* if, whenever A is a subformula of B, if $B \in \Psi$ then $A \in \Psi$.

1.7.1 Neighborhood filtration

Let $M = \langle W, \mathcal{N}, V\rangle$ be a neighborhood model and Ψ a set of formulas closed under subformulas. We define a binary relation $\equiv_\Psi$ in W:

$$w \equiv_\Psi x \quad \text{iff, for all } A \in \Psi,\ \ M, w \models A \text{ iff } M, x \models A.$$

(When it is clear from the context what Ψ is referred to, the subscript will sometimes be omitted.) $\equiv$ is an equivalence relation. If $w \in W$, then $[w]$ will denote the equivalence class of w. Define

$$W' \ = \ \{\,[w] \ \mid \ w \in W\,\};$$

and let $\mathcal{N}'$ be any function on W' such that, whenever $\Box A \in \Psi$ and $w \in W$,

$$\|A\|^M \in \mathcal{N}_w \ \text{ iff } \ \|A\|^{M'} \in \mathcal{N}'_w;$$

and let V' be any function on Var such that

$$V'(p_n) \;=\; \{\,[w] \;\mid\; w \in V(p_n)\,\},$$

for each $n \in \mathbb{N}$ such that $p_n \in \Psi$. It is clear that these conditions are unambiguous and that functions $\mathcal{N}'$ and V' always exist. If $M' = \langle W', \mathcal{N}', V \rangle$ then M' is called a (*neighborhood*) *filtration of M though Ψ.*

Theorem 1.7.1 (Filtration theorem (for neighborhood semantics))**.** *Suppose that Ψ, M, and M' are as above. Then for all $A \in \Psi$ and all $w \in W$,*

$$M, w \models A \quad \textit{if and only if} \quad M', [w] \models A.$$

Thus M and M' are equivalent modulo Ψ.

The proof – by induction on the length of A – is omitted.

Corollary 1.7.2. *The logic* **E** *has the finite model property.*

Proof. Suppose that A is not a theorem of **E**. By the Fundamental Theorem, A fails somewhere in the canonical model $\mathfrak{M}_{\mathbf{E}}$ for **E**. Let Ψ be the set of subformulas of A. By Theorem 1.7.1, A is rejected by every $M'_{\mathbf{E}}$ filtration of $\mathfrak{M}_{\mathbf{E}}$ through Ψ. Since Ψ is finite, $M'_{\mathbf{E}}$ is also finite; in fact, if Ψ has m elements, then $M'_{\mathbf{E}}$ has $\leqslant 2^m$ elements. $\square$

Corollary 1.7.3. *The logic* **E** *is decidable.*

Proof. Follows from 1.3.9 and 1.7.2. $\square$

That all systems mentioned in Section 1.4 have the f.m.p., and hence are decidable, may be proved similarly (that is, using the known completeness theorems in conjunction with Theorems 1.7.1 and 1.3.9). The same approach could be tried on the systems of Section 1.5, but the neighborhood semantics becomes cumbersome when iterated modalities are involved. As most of this dissertation deals with systems that are best studied by means of relational semantics, we shall now also introduce relational counterparts of the neighborhood filtrations.

1.7.2 Relational filtration

Suppose that $M_0 = \langle W, \mathcal{N}, V \rangle$ is an augmented neighborhood model and $M'_0 = \langle W', \mathcal{N}', V \rangle$ is the neighborhood filtration of M through some set Ψ of formulas closed under subformulas. We shall assume that $\Box\top \in \Psi$ if M_0 is not normal. Consider the relational model $M = \langle W, R, Q, V \rangle$ corresponding to M_0. Recall that, for all $w, x \in W$,

$$\text{if } w \notin Q \text{, then } w\,R\,x \text{ iff } x \in \bigcap \mathcal{N}_w;$$
$$w \in Q \text{ iff } \mathcal{N}_w = \varnothing.$$

When we look for a relational model $M' = \langle W', R', Q', V' \rangle$ to stand in the same relation to M'_0 as M does to M_0, we must keep in mind the requirement that, for all A such that $\Box A \in \Psi$,

$$\|A\|^M \in \mathcal{N}_w \;\text{ iff }\; \|A\|^{M_0} \in \mathcal{N}'_{[w]}, \quad \text{for all } w \in W.$$

Evidently, R', Q', and V' must fulfill the conditions:

i. $x \in \bigcap \mathcal{N}_w$ implies $[w]\,R'\,[x]$;
ii. $[w]\,R'\,[x]$ implies $[x] \in \bigcap \mathcal{N}'_{[w]}$;
iii. $[w] \in Q'$ iff $\mathcal{N}_w = \varnothing$;
iv. $V'(p_n) = \{\,[w] \mid w \in V(p_n)\,\}$, for each $n \in \mathbb{N}$ such that $p_n \in \Psi$.

In other words,

i'. $w\,R\,x$ implies $[w]\,R'\,[x]$;
ii'. $[w]\,R'\,[x]$ implies that for all A such that $\Box A \in \Psi$, if $M, w \models \Box A$ then $M, x \models A$;
iii'. $Q' \;=\; \{\,[w] \mid w \in Q\,\}$.

Thus, if $M = \langle W, R, Q, V\rangle$ is a relational model, Ψ is closed under subformulas, and $M' = \langle W', R', Q', V'\rangle$ is a relational model satisfying conditions (i'), (ii'), (iii'), and (iv), then we say that M' is a (*relational*) *filtration of M through Ψ*. Note that W' and Q', but not R' and V', are uniquely determined by M and Ψ.

Theorem 1.7.4 (Filtration theorem (for neighborhood semantics))**.** *Suppose that Ψ, M, and M' are as above (thus $\Box\top \in \Psi$ if M is not normal). Then for all $A \in \Psi$ and all $w \in W$,*

$$M, w \models A \quad \textit{if and only if} \quad M', [w] \models A.$$

Hence M and M' are equivalent modulo Ψ.

Suppose $M' = \langle W', R', Q', V'\rangle$ is a filtration of a relational model M through some set Ψ closed under subformulas (thus $\Box\top \in \Psi$ if M is not normal). If it holds that

$$[w]\, R'\, [x] \text{ iff, for all } A \text{ such that } \Box A \in \Psi, \text{ if } M, w \models \Box A \text{ then } M, x \models A;$$

then we shall say that M' is a *coarsest* filtration of M through Ψ. If instead it holds that, for every filtration $\langle W', R'', Q', V'\rangle$ through Ψ, $R' \subseteq R''$, then M' is said to be a *finest* filtration of M through Ψ. Note that M' is a finest filtration of M through Ψ iff whenever $[w]R'[x]$, there are $w' \equiv w$ and $x' \equiv x$ such that $w'\, R\, x'$.

Theorem 1.7.5. *Suppose L is a regular logic and that $\mathfrak{M}_L = \langle W_L, R_L, Q_L, V_L\rangle$ is the canonical model for L. Suppose Ψ a set of formulas closed under subformulas and containing $\Box\top$ if L is not normal. Then $M' = \langle W', R', Q', V'\rangle$ is a filtration of $\mathfrak{M}_L$ through Ψ only if it is a finest such filtration.*

Proof. Say that the filtration M' is a model for L. Take any $[w], [x] \in W'$ such that $[w]R'[x]$. Consider the sets

$$\begin{aligned} w' &= \{ A \mid M', [w] \models A \}; \\ x' &= \{ A \mid M', [x] \models A \}. \end{aligned}$$

Since M' is a model for L, the sets w' and x' must be L-maximal, and therefore, $w', x' \in W_L$. It then follows from Theorem 1.7.4 that $w' \in [w]$ and $x' \in [x]$. So, $w' \equiv w$ and $x' \equiv x$. It remains to show that $w'\, R_L\, x'$. Assume, for any A, that $\Box A \in w'$. Then $M', [w] \models \Box A$. As $[w]R'[x]$, it follows that $M', [x] \models A$. Thus $A \in x'$. Hence $w'\, R_L\, x'$. $\Box$

This result, which generalizes a theorem in [Segerberg, 1968b], is very plausible in view of the relationship between relational and neighborhood models. It shows that if we are interested in finding models of logics by way of filtrations, it suffices to consider finest filtrations. However, coarsest filtrations are easier to work with; usually we work with coarsest filtrations and find, in the end, that they are also finest.

It would be interesting to know whether every finite distinguishable model of a logic is a filtration through some set of formulas of the canonical models, or of some point generated submodel thereof.

1.7.3 Lemmon filtration

One particularly important kind of relational filtrations is encountered in connection with transitive models. Suppose $M = \langle W, R, Q, V\rangle$ is a transitive relational model, and suppose Ψ is a set of formulas closed under subformulas containing $\Box\top$ if M is not normal. Then a filtration $M' = \langle W', R', Q', V'\rangle$ of M through Ψ is called a *Lemmon filtration* of M through Ψ if R' satisfies this condition:

$$[w]R'[x] \text{ iff, for all } A, \text{ if } \Box A \in \Psi \text{ then } M', [w] \models \Box A \text{ implies that } M, [x] \models \Box A \text{ and } M, [x] \models A.$$

It is readily seen that if M is transitive, then Lemmon filtrations of It through exist. (The term "Lemmon filtration" is suggested in order to honor E. J. Lemmon, who was the first to study this condition; see [Lemmon:Scott, 1966].)

Theorem 1.7.6. *Suppose L is a normal extension of* **K4**. *Let $M = \langle W, R, V\rangle$ be any point generated submodel of the canonical model for L. Let Ψ be any set of formulas closed under subformulas. Suppose $M' = \langle W', R', V'\rangle$ is a Lemmon filtration of M through Ψ. Then for each particular logic S mentioned in Theorem 1.5.2, if $S \subseteq L$ then M' satisfies the condition linked to S in that theorem.*

If Ψ is logically finite, then M' is of course finite in addition to satisfying the condition mentioned.

Proof. It is easy to see that since M is point generated, so is M'. We give a proof for two cases as examples.

(iv) $S = \mathbf{K4.2}$. To show that M' is convergent, let $[w], [x]$ be distinct elements in W'. We may assume without loss of generality that $w, x \in W$. Clearly w and x must be distinct. Since, by Theorem 1.5.2, R is convergent, there must exist y such that wRy and xRy. Consequently, $[w]R'[y]$ and $[x]R'[y]$.

(xi) $S = \mathbf{K4B}$. To show that M' is symmetric, take any $[w], [x] \in W'$ such that $[w]R'[x]$. Again we may assume that $w, x \in W$. Suppose t is a generator of M. Then, since R is transitive, $t = w$ or tRw, and $t = x$ or tRx. If tRx then xRt – for, by Theorem 1.5.2, R is symmetric – so whether $t = w$ or tRw, we obtain xRw and hence $[x]R'[w]$. If $t = w$, then $w = x$, and hence immediately $[x]R'[w]$. $\square$

1.7.4 Closure under Boolean compounds

We end our discussion of filtrations with one last remark. Let Ψ be any set of formulas closed under subformulas. Let $B(\Psi)$ be the smallest set Θ such that

i. $\Psi \subseteq \Theta$;
ii. $\perp \in \Theta$;
iii. if $A, B \in \Theta$ then $(A \to B) \in \Theta$.

We say that $B(\Psi)$ is the *closure of* Ψ *under Boolean compounds.*

Theorem 1.7.7. *Suppose M' is a filtration (neighborhood or relational) of some model M through Ψ, where Ψ is a set of formulas closed under subformulas and containing $\Box\top$ if M is a nonnormal relational model. Then, for all formulas $A \in B(\Psi)$,*

$$M, w \models A \quad \textit{if and only if} \quad M', [w] \models A.$$

This sharpens the Filtration Theorem just a little: the models M and M' are equivalent modulo $B(\Psi)$. The obvious proof is omitted.

1.8 Historical remarks

There does not exist any account of the basic theory for classical logics as extensive as that given in this chapter; yet most of the material is not original with the author. In addition to the references in the text, some remarks on the history of the subject are in order.

It is unclear to this author who deserves the credit for having introduced classical logics. They are discussed, as a special case, in [Cresswell, 1971] under the name of "strongly classical intensional logics", and the Fundamental Theorem for classical logics is proved there. The mimeographed research report [Gabbay, 1969], which deals in part with classical modal predicate logics, also contains the Fundamental Theorem, as well as completeness results for the extensions of the logic **E** definable in terms of the schemata K, T, and the rules RR and RN. The connection between neighborhood and relational semantics is also noted there.

Gabbay attributes neighborhood semantics to Montague; it appears as a special case in [Montague, 1968]. Neighborhood semantics also appears in [Scott, 1968]. This author can testify that Scott knew about neighborhood semantics and had the proof of the Fundamental Theorem at least as early as in the spring of 1967. According to Scott, the idea of neighborhood semantics is implicit in older work of McKinsey and Tarski who therefore would be regarded as the true creditors. However that is, it seems that the importance of this kind of semantics was realized first by Montague and Scott.

The history and recent development of normal logics is well known. Regular logics were first studied in [Lemmon, 1957]; see also [Lemmon, 1966a], [Lemmon, 1966b]. (The term "regular" is ours.) The Fundamental Theorem for normal logics was first proved in [Lemmon:Scott, 1966], and [Makinson, 1966a] contains a very similar result. The basic idea, that of applying the method of Henkin's completeness proof for ordinary predicate logic to modal logic, also occurred independently to some other authors; see for example [Cresswell, 1967] and [Schutte, 1968].

Chapter 2

Normal systems

A longer but more adequate title for this Chapter would have been something like "Topics in the theory for normal systems", for the scope of our interest will be somewhat restricted: the whole chapter will be devoted to systems determined by classes of transitive frames. (In this chapter, by "frame" we always mean relational normal frame.)

There are at least two reasons why one might argue that transitive frames are particularly interesting. One is that a good portion of the logics traditionally thought to be important are in fact characterized by transitive frames; **S4** and **S5** and all their normal extensions belong in this category. Another reason is that transitive frames are pleasant to work with. This argument is not as irresponsible as it perhaps sounds. Clearly it is valuable if we are able to give a fruitful analysis of an important fragment of modal logics; it may lead us on to analyses of other kinds of systems. And from the point of view of their amenability to analysis, the logics determined by classes of normal transitive frames seem to me to be a most natural starting point.

2.1 Clusters

Whenever $F = \langle W, R \rangle$ is a transitive frame, $\sim$ will denote the following binary relation on W:

$$x \sim y \text{ iff either } xRy \text{ and } yRx\text{, or } x = y.$$

It is easy to see that $\sim$ is always an equivalence relation on W. If $w \in W$, $w/\!\sim$ will denote the equivalence class of w under $\sim$. An equivalence class under $\sim$ is called a *cluster*. We distinguish three kinds of clusters:

i. *proper* clusters – those with at least two elements;
ii. *simple* clusters – those with exactly one element, which is reflexive;
iii. *degenerate* clusters — those with exactly one element, which is irreflexive.

(An element w is *reflexive* in a frame $\langle W, R \rangle$ if wRw, *irreflexive* otherwize.) The first two kinds of clusters are also called *nondegenerate*. It is immediate that every element of a proper cluster is reflexive; thus only in a degenerate cluster can one find an irreflexive element. (The term "cluster" was first used in [Segerberg, 1968b]. The reader should beware that the present concept of cluster differs slightly from the older one.)

Let x, y be elements in a transitive frame $F = \langle W, R \rangle$. We say that x *precedes* y and y *succeeds* x if xRy and $x \not\sim y$. If $X \subseteq W$, we say that x *precedes* (*succeeds*) X if x precedes (succeeds) every element of X. If $Y \subseteq W$, we say that X *precedes* (*succeeds*) Y if every element of X precedes (succeeds) every element of Y. Thus precedence and succession are irreflexive and transitive relations. An element is *final* (*initial*) if it is not succeeded (preceded) by any element in W. It is *last* (*first*) if it succeeds (precedes) every other element in W. Similarly, a cluster in F is *final* (*initial*) if it is not succeeded (preceded) by any element in W, *last* (*first*) if it succeeds (precedes) every element in W that is not in the cluster. Thus, if a frame has a last (first) cluster, it is unique. We say that x *immediately precedes* (*immediately succeeds*) y if x precedes (succeeds) y and there is no element z such that x precedes (succeeds) z and z precedes (succeeds) y. Similarly, a cluster C *immediately precedes* (*immediately succeeds*) a cluster D if C precedes (succeeds) D and there is no cluster E such that C precedes (succeeds) E and E precedes (succeeds) D.

Clusters are important if one wants to study partially or strict partially ordered frames. For notice that if F is a transitive frame, then

- F is partially ordered iff every cluster in F is simple;
- F is strict partially ordered iff every cluster in F is degenerate.

(A relation is a *partial ordering* if it is reflexive, transitive, and antisymmetric; a *strict partial ordering* if it is irreflexive and transitive.) Using Theorem 1.7.6 for the completeness part, we note that the following logics are determined by finiteness and transitivity in conjunction with one condition in terms of clusters:

K4 – (no other condition);
D4 – no final cluster is degenerate;
S4 – no cluster is degenerate;
K4.2 – there is a last cluster;
D4.2 – there is a last cluster, and it is not degenerate;
S4.2 – no cluster is degenerate, and there is a last cluster;
K4.3 – of every two distinct clusters, one precedes the other;
D4.3 – of every two distinct clusters, one precedes the other, and the last cluster of the frame is not degenerate;
S4.3 – of every two distinct clusters, one precedes the other, and no cluster is degenerate;
K4E – either there in only one cluster, or there are only two clusters and they are adjacent, the first degenerate, the last nondegenerate;
K4B – there is only one cluster;
D4E – either there is only one cluster and it is nondegenerate, or there are only two clusters and they are adjacent, the first degenerate, the last nondegenerate;
S5 – there is only one cluster, and it is nondegenerate.

2.1.1 "Bulldozing" clusters in transitive frames

We shall now explain a technique that will be used a good deal in this chapter. By means of it a reflexive transitive model can be transformed into an equivalent partially ordered model, and a transitive model, not necessarily reflexive, can be transformed into an equivalent strict partially ordered model. Thus in the former case all proper clusters, and in the latter case all nondegenerate clusters, can be "flattened out" without disturbing the truth conditions in the models. We have decided to call this technique "bulldozing"; this is perhaps a term one would not expect to encounter in an essay on pure logic, but it is a fairly adequate one, the best we have been able to think of.

Suppose $M = \langle W, R, V\rangle$ is a transitive [reflexive and transitive] model. (We describe the transitive / strict partial ordering case; within square brackets we give the changes that would yield the reflexive transitive / partial ordering case.) Let each nondegenerate cluster C of M be embedded in a set C^+ in such a way that if C and D are distinct clusters then C^+ and D^+ have no common element. For each C^+ fix a strict linear [linear] ordering R^+ of C^+, define

$$C^* \;=\; \{\, \langle c, i\rangle \mid c \in C^+ \text{ and } i \in \mathbb{N}\,\}.$$

There is no loss of generality if we assume that W and the C^*'s are disjoint. Let

$$W' \;=\; \Big(W \;\setminus\; \bigcup\{\, C \mid C \text{ is a nondegenerate cluster in } M\,\}\Big) \\ \cup\; \Big(\bigcup\{\, C^+ \mid C \text{ is a nondegenerate cluster in } M\,\}\Big).$$

In each nondegenerate cluster C we arbitrarily select some element which we denote by $\gamma(C)$. The following then defines a function $f\colon W' \to W$:

$$f(\xi) \;=\; \begin{cases} \xi, & \text{if } \xi \in W; \\ c, & \text{if } \xi = \langle c, i\rangle \text{ and } c \in W; \\ \gamma(c/\!\sim), & \text{if } \xi = \langle c, i\rangle \text{ and } c \notin W. \end{cases}$$

We define the following relation on W':

$$\xi\, R'\, \eta \ \text{ iff } \quad \text{either (1) } \xi \in W \text{ or } \eta \in W, \text{ and } \xi\, R\, \eta; \\ \text{or (2) } \xi = \langle c, i\rangle,\ \eta = \langle d, j\rangle \text{ and } \ \text{either (i) } c \not\sim d \text{ and } \gamma(c/{\sim})\, R\, \gamma(d/{\sim}), \\ \text{or (ii) } c \sim d \text{ and } i < j, \\ \text{or (iii) } c \sim d \text{ and } i = j \text{ and } c\, R^+\, d.$$

Finally we stipulate, for every $n \in \mathbb{N}$,

$$V'(p_n) \ = \ \{\, \xi \in W' \ \mid \ f(\xi) \in V(p_n)\,\}.$$

Let $M' = \langle W', R', V'\rangle$. It is easy to see that f is a p-morphism from M' to M (for if $\xi R' \eta$ then certainly $f(\xi) R f(\eta)$; and if $f(\xi) R f(\eta)$ then either $f(\xi)$ and $f(\eta)$ belong to distinct clusters and hence $\xi R' \eta$, or $f(\xi) \sim f(\eta)$ in which case the cluster $f(\xi)/{\sim} = f(\eta)/{\sim}$ is nondegenerate, so $f(\langle f(\eta), j\rangle) = f(\eta)$ and $\xi\, R'\, \langle f(\eta), j\rangle$ for sufficiently large j). Moreover, f is reliable on every propositional letter. Hence, by the P-Morphism Theorem, M and M' are equivalent. Note that this result holds no matter how the embeddings C^+ are chosen; in particular, it is not excluded that $C = C+$. It is clear that M' is irreflexive [reflexive and antisymmetric] as well as transitive. Hence we have proved:

Theorem 2.1.1 (The Bulldozer Theorem)**.** *For every transitive [reflexive and transitive] model there exists an equivalent strict partially [partially] ordered model.*

Actually there is more in the proof of the Bulldozer Theorem then stated under 2.1.1. Informally our construction, transforming M into M', might be described as follows: Each nondegenerate cluster C is replaced by a sequence of ω distinct copies of C, each of which forms a strict linear [linear] order and behaves like its prototype C with respect to the environment outside C. Note that in the reflexive case it would be possible to leave simple clusters alone and only "bulldoze" the proper clusters.

We note the following important corollary:

Theorem 2.1.2. *For every transitive connected [reflexive transitive connected] model there is an equivalent strict linearly [linearly] ordered model.*

Proof. The R^+'s in the construction described above are strict linear [linear] orderings. □

The completeness results mentioned earlier in this section are easily improved in the light of the Bulldozer Theorem and its corollary. For example, the following six logics are determined by the following conditions on frames:

K4 – strict partial ordering;
D4 – strict partial ordering, no final element;
S4 – partial ordering;
K4.3 – strict linear ordering;
D4.3 – strict linear ordering, no last element;
S4.3 – linear ordering.

2.1.2 Completeness for trees

Before closing this section we shall mention yet some other completeness result for **K4**, **D4**, and **S4**, namely in terms of trees. By a *tree* we understand a structure $\mathcal{T} = \langle T, t, R\rangle$ such that T is a set, $t \in T$, R is a binary relation on T, and the following conditions are satisfied:

i. if $w\, R\, t$ then $w = t$;
ii. if $w \neq t$ then there exists exactly one $x \neq w$ such $x\, R\, w$;
iii. if $w \neq t$ then, for some $x_0, x_1, \ldots, x_n$, (where $n \geqslant 1$), we have $t = x_0$, $w = x_n$, and $x_0\, R\, x_1$, $x_1\, R\, x_2$, $\ldots$, $x_{n-1}\, R\, x_n$.

We shall say that the tree $\mathcal{T}$ is *reflexive* or *irreflexive* according to whether R is reflexive or irreflexive. We shall say, somewhat carelessly, that a frame $F = \langle W, R\rangle$ is a tree if F is generated by some element t and there is a binary relation S on W such that $\langle W, t, S\rangle$ is a tree and $S^+ = R$. (S^+ is the *ancestral* of S, that is $x\, S^+\, y$ iff there are $x_0, \ldots, x_n$ ($n \geqslant 1$) such that $x_0 = x$, $x_n = y$, and $x_0\, S\, x_1$, $x_1\, S\, x_2$, $\ldots$, $x_{n-1}\, S\, x_n$.)

It was shown in [Kripke, 1963] that **S4** is determined by the class of reflexive trees. Similarly, **K4** is determined by the class of irreflexive trees, **D4** by the class of infinite irreflexive trees. We shall not reproduce Kripke's proof here, nor shall we prove the latter contention. (Kripke's proof is long, and the author's proofs are quite messy. In fact, there do not seem to exist any simple proofs in the literature. We therefore suggest this as a research problem: to find simple proofs of the last three completeness results.)

There exist many other completeness results for the logics mentioned in this section; see [Bull, 1965] and [Segerberg, 1971].

2.2 Strict partial orderings

In this section we shall examine the following two schemata:

W. $\Box(\Box A \to A) \to \Box A$
Z. $\Box(\Box A \to A) \to (\Diamond\Box A \to \Box A)$

To find completeness results for logics defined in terms of these schemata is considerably more difficult than for any of the systems we have dealt with up to now. Every schema that is derivable in a logic has some effect on the canonical model of that logic; but it is not easy to see what effect W or Z has on canonical models. Our solution to this problem is to use filtration theory.

Lemma 2.2.1. *Suppose L is a normal extension of* **K4Z**. *Let $\mathfrak{M} = \langle W, R, V\rangle$ be a point generated submodel of the canonical model for L, and let $\mathfrak{M}' = \langle W', R', V'\rangle$ be the Lemmon filtration of $\mathfrak{M}$ through some logically finite set Ψ of formulas closed under subformulas. Then, for every cluster C in $\mathfrak{M}'$ that is not last in $\mathfrak{M}'$, there exists some irreflexive element that is final in $\{w \mid [w] \in C\}$.*

Proof. Suppose C is a cluster in $\mathfrak{M}'$ that is not last. If C is degenerate, the conclusion of the lemma is obviously true. Assume therefore that C is not degenerate. Since Ψ is logically finite, $\mathfrak{M}'$ is a finite distinguishable model. For each $\xi \notin C$ we define a formula A_ξ in the following manner. Let $\eta_1, \ldots \eta_k$ be all the elements of C. Then there exist formulas $A_{\xi,1}, \ldots, A_{\xi,k} \in \Psi$ such that they are all true at ξ, but $A_{\xi,1}$ fails at η_1, $\ldots$, and $A_{\xi,k}$ fails at η_k. Define

$$A_\xi \;=\; A_{\xi,1} \wedge \ldots \wedge A_{\xi,k}.$$

Let A be the disjunction of all the A_ξ's, where $\xi \notin C$ (in some order). Note that $A \in B(\Psi)$; that is, A is a Boolean combination of formulas in Ψ. Then by Theorem 1.7.7 it holds that for all $u \in W$

(1) $\mathfrak{M}, u \models A$ iff $[u] \notin C$.

Take any $v \in W$ such that $[v] \in C$. By (1),

(2) $\mathfrak{M}, v \not\models A$.

Since C is not last in $\mathfrak{M}'$, there exists some $\xi \notin C$ such that ξ does not precede C. Take any $x \in \xi$. If $x\, R\, y$ then $\xi\, R'\, [y]$ and hence $[y] \notin C$. By (1), $\mathfrak{M}, y \models A$, and therefore

(3) $\mathfrak{M}, x \models \Box A$.

The model $\mathfrak{M}$ is point generated; let t be an element generating $\mathfrak{M}$. It follows from (3) that

(4) $\mathfrak{M}, t \models \Diamond\Box A$.

and it follows from (2) that

(5) $\mathfrak{M}, t \not\models \Box A$.

Since we have assumed that $\mathbf{K4Z} \subseteq L$, (4) and (5) imply that

(6) $\mathfrak{M}, t \not\models \Box(\Box A \to A)$.

Consequently, there exists some w such that $\mathfrak{M}, w \models \Box A$, but $\mathfrak{M}, w \not\models A$. Clearly, w is irreflexive. By (1), $[w] \notin C$. Suppose there exists w' such that $w\, R\, w'$. Then $\mathfrak{M}, w' \models A$, so by (1) $[w'] \notin C$. Thus w is final in $\{y \mid [y] \in C\}$. $\Box$

2.2.1 The schema W

Theorem 2.2.2. *The logic* **K4W** *is determined by the class of all finite strict partial orderings.*

Here the word "finite" is essential.

Proof. There is no question about the consistency part. Suppose now that a formula A_0 is not derivable in **K4W**. Then, by the Fundamental Theorem and the Generation Theorem, A_0 is rejected by a certain point generated submodel $\mathfrak{M} = \langle W, R, V\rangle$ of the canonical model $\mathfrak{M}_{\mathbf{K4W}}$. Let Ψ be the smallest set containing each subformula of A_0 and of $\Box\bot$. Let $\mathfrak{M}' = \langle W', R', V'\rangle$ be the Lemmon filtration of $\mathfrak{M}$ through Ψ. By the Filtration Theorem, A_0 is rejected by $\mathfrak{M}'$, and $\mathfrak{M}'$ is point generated, finite, and transitive. $\mathfrak{M}'$ would then be a strict partial ordering if it contained only degenerate clusters. In the general case, however, $\mathfrak{M}'$ will contain also nondegenerate clusters, and it is our problem to show that they can be "removed".

First, we notice that $\mathfrak{M}'$ contains no nondegenerate last cluster, in fact, no nondegenerate final clusters. For suppose C would be such a cluster: pick any $\xi \in C$. Since C is nondegenerate, ξ is reflexive. Hence $\mathfrak{M}', \xi \not\models \Box\bot$, so by the Filtration Theorem (recall that $\Box\bot \in \Psi$!), if $x \in \xi$, then $\mathfrak{M}, x \not\models \Box\bot$. Since every instance of the schema W holds at every point in $\mathfrak{M}$, it follows that $\mathfrak{M}, x \not\models \Box(\Box\bot \to \bot)$. Hence there exists a point y such that $x\,R\,y$ and $\mathfrak{M}, y \models \Box\bot$. By the Filtration Theorem, $\mathfrak{M}, [y] \models \Box\bot$, so $[y]\,R'\,[x]$ is not possible. However, $x\,R\,y$ implies that $[x]\,R'\,[y]$, which means that $[y]$ is a successor of C. This contradicts the finality of C.

For every nondegenerate cluster C in $\mathfrak{M}'$, let γ_C designate an irreflexive element final in $\{w \mid [w] \in C\}$ – since W implies Z and, as we have just seen, every nondegenerate cluster in $\mathfrak{M}'$ is nonfinal, Lemma 2.2.1 guarantees that the definition of γ is meaningful. Let R_C be an arbitrary ordering of C such that γ_C becomes last in C. We define the following binary relation on W':

$$\xi\,R^\star\,\eta \quad \text{iff} \quad \begin{array}{l} \text{either (1) } \xi \text{ and } \eta \text{ do not belong to the same cluster in } \mathfrak{M}', \text{ and } \xi\,R'\,\eta, \\ \text{or (2) } \xi \text{ and } \eta \text{ belong to the same cluster } C \text{ in } \mathfrak{M}', \text{ and } \xi\,R_C\,\eta. \end{array}$$

Let $\mathfrak{M}^\star = \langle W', R^\star, V'\rangle$. Then $\mathfrak{M}'$ and $\mathfrak{M}^\star$ are equivalent modulo Ψ. In fact, for all $A \in \Psi$ and all $\xi \in W'$,

$$\mathfrak{M}', \xi \models A \quad \text{if and only if} \quad \mathfrak{M}^\star, \xi \models A.$$

The proof, by induction on the length of A, is trivial except in the following place in the induction step. Suppose that $\Box B \in \Psi$ and the assertion holds for B, and suppose that C is a nondegenerate cluster and that $\mathfrak{M}', \xi \not\models \Box B$, for a certain $\xi \in C$. Then $\mathfrak{M}', [\gamma_C] \not\models \Box B$, and hence by the Filtration Theorem, $\mathfrak{M}, \gamma_C \not\models \Box B$. Hence there exists some z such that $\gamma_C\,R\,z$ and $\mathfrak{M}, z \not\models B$. By the Filtration Theorem again, $\mathfrak{M}', [z] \not\models B$, so by induction hypothesis, $\mathfrak{M}^\star, [z] \not\models B$. But γ_C is irreflexive and last in $\{w \mid [w] \in C\}$. Therefore, $[z]$ succeeds C in $\mathfrak{M}'$. Hence $\xi\,R^\star\,[z]$, and so $\mathfrak{M}^\star, \xi \not\models \Box B$.

Thus A_0, the formula that was nod derivable in **K4W**, fails in $\mathfrak{M}^\star$. As $\mathfrak{M}^\star$ is a finite strict partial ordering, the proof is complete. □

Corollary 2.2.3. *The logic* **K4W** *is determined by the class of finite irreflexive trees.*

Proof. It is easy to see that each finite strict partially ordered model can be "untangled" with no harm done to the truth conditions. (To describe this procedure formally and in detail is a bit messy, just as in the case of **K4**, **D4**, and **S4** above, and is not rewarding in any corresponding measure.) □

Corollary 2.2.4. *The logic* **K4.3W** *is determined by the class of finite strict linear orderings.*

Proof. Each R_C in the proof of Theorem 2.2.2 is a strict linear ordering. □

Corollary 2.2.5. *The logic* **K4.3W** *is determined by the single frame* $\langle \mathbb{N}, >\rangle$.

Proof. Every finite strict linear ordering is isomorphic to some point generated subframe of $\langle \mathbb{N}, >\rangle$. □

2.2.2 The schema Z

However, we shall now go on to analyse the schema Z. It derives its main interest when joined to **K4.3**, but it is instructive to study it already in the field of **K4**. We introduce the following vocabulary. A frame $F = \langle W, R\rangle$ is called a *kite* if R is transitive and anti-symmetric, and there exists some $z \in W$ such that

i. $w \in W$ iff $w\,R\,z$ or $w = z$ or $z\,R\,w$;
ii. the set $\{w \mid w\,R\,z\}$ is finite;
iii. the set $\{w \mid z\,R\,w\}$ is connected under R and of order type $\leqslant \omega$.

Note that connected kites are all tails. Reflexive kites, and infinite irreflexive kites, are always convergent. A kite is reflexive or irreflexive when ifs alternative relation is.

Theorem 2.2.6. *The logic* **K4Z** *is determined by the class consisting of all finite irreflexive trees and all irreflexive kites.*

Proof. We begin by proving the consistency part. Suppose that $M = \langle W, R, V\rangle$ is an irreflexive transitive model such that, for some formula A and some point $w \in W$,

(1) $M, w \models \Box(\Box A \to A)$;
(2) $M, w \models \Diamond\Box A$;
(3) $M, w \not\models \Box A$.

By (3), there exists some x_1 such that $w\,R\,x_1$ and $M, x_1 \not\models A$. Since R is irreflexive, $x_1 \neq w$. By (1), $M, x_1 \not\models \Box A$. Hence there exists some x_2 such that $x_1\,R\,x_2$ and $M, x_2 \not\models A$. Since R is irreflexive and transitive, $x_2 \neq x_1$ and $x_2 \neq w$. By (2), $M, x_2 \not\models \Box A$; etc. Hence

(4) there is an infinite sequence of distinct successive elements $w, x_1, x_2, \ldots$ at which both $\Box A$ and A are false.

Thus M is infinite and, in particular, not a finite tree. Suppose M is a kite. Let z be an element such as specified in the definition of kite. Then there exists $i \in \mathbb{N}$ such that for all $n \in \mathbb{N}$,

(5) if $i < n$ then $z\,R\,x_n$.

By (2), there is some point y such that

(6) $M, y \models \Box A$.

Since M is a kite, there exists $j \in \mathbb{N}$ such that for all $n \in \mathbb{N}$,

(7) if $j < n$ then $y\,R\,x_n$.

Take any k larger than both i and j. Then, by (6) and (7), $M, x_k \models A$. However, by (4) and (5), $M, x_k \not\models A$. Thus the assumption that M is a kite leads to a contradiction. This ends the proof of the consistency part.

The proof of the completeness part we only sketch. Consider the situation in the proof of Theorem 2.2.2. It is described in detail there how to handle any nondegenerate cluster that is not last. If there is a last, nondegenerate cluster, the Bulldozer Theorem can be applied. Thus for any nontheorem of **K4Z**, one can always prove the existence of a counter-model is either a finite tree or a kite. $\square$

Theorem 2.2.7. *The logic* **D4Z** *is determined by the class of infinite irreflexive kites.*

Proof. Soundness is clear. For completeness, we first prove that the scheme G, which is $\Diamond\Box A \to \Box\Diamond A$, is derivable in **D4Z**. Let A be any formula. Then
(a) $\vdash \Box A \to \Diamond A$ (because of D), hence
(b) $\vdash \Box\Box A \to \Box\Diamond A$ (rule RN), hence
(c) $\vdash \Box A \to \Box\Diamond A$ (schema 4 and TF), hence
(d) $\vdash \Diamond\Box A \to \Diamond\Box\Diamond\Box A$ (TF and RN). Now,
(e) $\vdash \Box(\Box\Diamond A \to \Diamond A) \to (\Diamond\Box\Diamond A \to \Box\Diamond A)$ (because of Z). But
(f) $\vdash \Box\Diamond A \to \Diamond\Diamond A$ (because of D) and

(g) $\vdash \Diamond\Diamond A \to \Diamond A$ (because of schema 4), so
(h) $\vdash \Box\Diamond A \to \Diamond A$ and hence
(i) $\vdash \Box(\Box\Diamond A \to \Diamond A)$. This and (e) yields
(j) $\vdash \Diamond\Box\Diamond A \to \Box\Diamond A$. Finally, this and (d) give us
(k) $\vdash \Diamond\Box A \to \Box\Diamond A$, which is what we wanted.

Consider now the proof of Theorem 2.2.2: in the present case, mutatis mutandis, the filtration $\mathfrak{M}'$ will be convergent, because of G, and hence contain a last cluster. Because of D, this cluster will be nondegenerate. From this it follows that $\mathfrak{M}^\star$ will be an infinite kite. □

Corollary 2.2.8. *The logic* **K4.3Z** *is determined by the class of strict linear orderings of type* $\leqslant \omega$.

Proof. See the proof of Corollary 2.2.4. □

Corollary 2.2.9. *The logic* **D4.3Z** *is determined by the single frame* $\langle \mathbb{N}, < \rangle$.

2.2.3 The schema W_0

One formula that is of interest in this connection is

W_0. $\Box\Diamond\top \to \Box\bot$.

It should be noted that W_0 is an instance of W. Let us call a point in a frame a *dead end* if it completely lacks alternatives. Suppose that L is a normal extension of $\mathbf{KW_0}$. Then it is easy to prove that every element in the canonical model $\mathfrak{M}_L$ either is a dead end or has an alternative that is a dead end. (This result is particularly plausible if W_0 is written on the equivalent form $\Box\bot \vee \Diamond\Box\bot$.) One may say that W_0 is exactly the difference between **K4W** and **K4Z**:

$$\mathbf{K4W} \;=\; \mathbf{K4ZW_0}.$$

W_0 has a special interest in connection with tense logic, as it "puts and end to time". For example, one may easily verify that $\mathbf{K4.3W_0}$ is determined by the class of all strict linear orderings that have a last element.

2.2.4 The logic of (finite) kites

It is natural to ask whether the logic determined by the class of all kites can be axiomatized. The answer is positive. One may begin by noting that, whereas finite kites are not convergent according to our definition of convergence (see Section 1.5), yet they are convergent in the following weaker sense: Let $\langle W, R\rangle$ be a frame. Then we shall say that R is *weakly convergent* if for all $x, y \in W$, if neither $x = y$ nor $x\,R\,y$ nor $y\,R\,x$, then for some z, both $x\,R\,z$ and $y\,R\,z$. Piecewise weak convergence is defined in the obvious way. Consider the special case of G_0:

G_1. $\Diamond(A \wedge \Box A) \to \Box(A \vee \Diamond A)$.

It is possible to show that for every normal extension L of $\mathbf{KG_1}$, the canonical model $\mathfrak{M}_L$ is piecewise weakly convergent. It is readily seen that every finite filtration of any point generated submodel of the canonical model of a normal logic at least as strong as $\mathbf{K4G_1}$ will have at most one final cluster; that is, the only final cluster will be the last one. This last cluster may or may not be degenerate. Thus we conclude:

Theorem 2.2.10. *The logic* $\mathbf{K4G_1Z}$ *is determined by the class of all kites.*

Adding W_0 to this logic to this logic has the effect that it always makes the last cluster just referred to degenerate. Thus we immediately obtain our last result in this section:

Theorem 2.2.11. *The logic* $\mathbf{K4G_1ZW_0}$ *is determined by the class of all finite kites.*

It should be noted that our proofs show that every logic treated has the f.m.p. (Warning: If a logic is determined by a class $\mathfrak{C}$ and has the f.m.p., it does *not* follow that the logic is determined by the class of finite frames in $\mathfrak{C}$. Theorem 2.2.7 affords a good counterexample.)

2.3 Partial orderings

The principal new schemata to be studied in this section – corresponding to W and Z of the previous section – are

Grz. $\Box(\Box(A \to \Box A) \to A) \to A$.

Dum. $\Box(\Box(A \to \Box A) \to A) \to (\Diamond\Box A \to A)$.

(The schemata are named in honor of Grzegorczyk and Dummett. Regarding the historical background, see the last section of this chapter.)

2.3.1 The schema Grz

We introduce a new definition. Suppose M' is the Lemmon filtration of some reflexive transitive model $M = \langle W, R, V\rangle$, and suppose that C is a cluster in M'. We shall say that an element $\xi \in C$ is *virtually last* in C if there is some $w \in \xi$ such that, for all $x \in W$, if $w\,R\,x$ and $[x] \in C$ then $\xi = [x] = [y]$. (The concept 'virtually last' is meaningful even if M is not reflexive, but that case is not considered here.) It is clear that simple clusters always have (unique!) virtually last elements. It is also clear that if M is connected, then if a proper cluster has a virtually last element, that element is unique.

Lemma 2.3.1. *Suppose L is a normal extension of* **S4Grz**. *Let $\mathfrak{M} = \langle W, R, V\rangle$ be a point generated submodel of the canonical model for L, and suppose $\mathfrak{M}' = \langle W', R', V'\rangle$ is the Lemmon filtration of $\mathfrak{M}$ through some logically finite set Ψ of formulas closed under subformulas. Then every proper cluster in $\mathfrak{M}'$ contains a virtually last element.*

Proof. Suppose that C is a proper cluster in $\mathfrak{M}'$. Let $\xi_1, \ldots, \xi_r$ be an enumeration of all elements of C; this is possible since Ψ is logically finite. We shall now inductively define a sequence $(\Gamma_n)_{n\in\mathbb{N}}$ of subsets of C, and a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of $\mathfrak{M}$.

Stage 0. Let $\Gamma_0 = C$. Pick any element in $\{w \mid [w] \in C\}$ and call it x_0.

Stage n+1.

Case 1. There is some i such that, for some y, these conditions hold:

$$\xi_i \in \Gamma_n; \qquad x_n\,R\,y; \qquad [y] \in C; \qquad \xi_i \notin \{\,[w] \mid y\,R\,w\,\}.$$

Let i_0 be the smallest such i and define $\Gamma_{n+1} = \Gamma_n \setminus \{\xi_{i_0}\}$.

As x_{n+1}, pick any element y that satisfies the above conditions with $i = i_0$.

Case 2. There is no such i as described in Case 1.

Then define $\Gamma_{n+1} = \Gamma_n$ and $x_{n+1} = x_n$.

We shall make a few observations in connection with this definition. For every n,

(1) if $i \leqslant j$, then $x_i\,R\,x_j$ and $\Gamma_i \supseteq \Gamma_j$.

This is obvious. Furthermore, for every n,

(2) $\{\,[y] \in C \mid x_n\,R\,y\,\} \subseteq \Gamma_n$.
For suppose $\eta \in C \setminus \Gamma_n$, for some n. Let j be the largest integer such that $\eta \in \Gamma_j$. Evidently $j < n$ and $\eta \notin \{\,[w] \mid x_{j+1}\,R\,w\,\}$. Take any y such that $x_n\,R\,y$ and $[y] \in C$. Then, since $x_{j+1}\,R\,x_n$, and R is transitive, we have $x_{j+1}\,R\,y$. Hence $\eta \neq [y]$.

Consequently, since R is reflexive, for every n, $[x_n] \in \Gamma_n$ and so $\Gamma_n \neq \varnothing$. Since C is finite, $\bigcap_{n\in\mathbb{N}} \Gamma_n \neq \varnothing$. Let m be the smallest integer such that

$$\Gamma_m \;=\; \textstyle\bigcap_{n\in\mathbb{N}} \Gamma_n.$$

We now continue the proof of the lemma. There are two cases to consider. First, the easy one: suppose that $\Gamma_m = \{\xi\}$, for some $\xi \in C$. Since $[x_m] \in \Gamma_m$ by the above, $x_m \in \xi$. Take any y such that $x_m \, R \, y$ and $[y] \in C$. By (2), $[y] \in \Gamma_m$. Hence $\xi = [y]$, and thus ξ is virtually last in C.

Now the other case: suppose Γ_m contains at least two elements. Notice that up to this point no use has been made of the important assumption that L contains all instances of the new schema Grz. This assumption will now be used to show that the present case cannot arise. Suppose ξ and η are distinct elements of Γ_m. Then, for all $y \in W$,

(3) if $x_m \, R \, y$ and $[y] \in C$, then $\xi, \eta \in \{\, [w] \mid y \, R \, w \,\}$.

Since $\xi \neq \eta$, there is some $A \in \Psi$ such that $\mathfrak{M}', \xi \models A$ and $\mathfrak{M}', \eta \not\models A$. Then, by the Filtration Theorem,

(4) $\mathfrak{M}, x \models A$ and $\mathfrak{M}, y \not\models A$,

for every $x \in \xi$ and every $y \in \eta$. By the same method as in the proof of Lemma 2.2.1, we can find a formula B that is a Boolean combination of formulas in Ψ and is such that, for all $\zeta \in W'$,

$$\mathfrak{M}', \zeta \models B \quad \text{if and only if} \quad \zeta \notin C.$$

Using Theorem 1.7.7 (about Boolean combinations of formulas from Ψ), we conclude that, for all $z \in W$,

(5) $\mathfrak{M}, z \models B$ if and only if $[z] \notin C$.

Pick any $y \in \eta$ such that $x_m \, R \, y$. It is clear, from (4) and (5), that $\mathfrak{M}, y \not\models A \vee B$. Hence, because every instance of Grz holds everywhere in $\mathfrak{M}$,

$$\mathfrak{M}, y \not\models \Box(\Box(A \vee B \to \Box(A \vee B)) \to A \vee B).$$

Consequently, there is some z such that $y \, R \, z$ and

(6) $\mathfrak{M}, z \models \Box(A \vee B \to \Box(A \vee B))$;
(7) $\mathfrak{M}, z \not\models A$;
(8) $\mathfrak{M}, z \not\models B$.

R is transitive, so $x_m \, R \, z$. From (5) and (8) it follows that $[z] \in C$. Thus, by (3), there exists some $x \in \xi$ such that $z \, R \, x$. From (4) and (6) it follows that

(9) $\mathfrak{M}, x \models \Box(A \vee B)$.

Since $x_m \, R \, x$ and $[x] \in C$, (3) implies that there exists $y' \in \eta$ such that $x \, R \, y'$. Then, by (9), $\mathfrak{M}, y' \models A \vee B$. However, this is in contradiction with (4) and (5).

The conclusion of all this is that Γ_m always contains just one element, and, as we saw, this element is virtually last in C. Thus the proof of Lemma 2.3.1 is complete. $\Box$

Theorem 2.3.2. *The logic* **S4Grz** *is determined by the class of all finite partial orderings.*

The word "finite" is essential here.

Proof. There is no question about soundness. In order to establish completeness, it will be enough to take Ψ, $\mathfrak{M}$, and $\mathfrak{M}'$ as in the preceding lemma and prove the existence of a partially ordered finite model $\mathfrak{M}''$ such that $\mathfrak{M}'$ and $\mathfrak{M}''$ are equivalent modulo Ψ. We know that $\mathfrak{M}'$ is finite point generated reflexive transitive. For every proper cluster C let γ_C be a virtually last element in C; the lemma guarantees that γ_C exists. Let $\leqslant_C$ bean arbitrary linear ordering of C under which γ_C is last. Define the following ordering on W':

$$\begin{aligned} \xi \, R'' \, \eta \quad \text{iff} \quad & \text{either (1) } \xi \text{ and } \eta \text{ belong to distinct clusters in } \mathfrak{M}' \text{ and } \xi \, R' \, \eta, \\ & \text{or } \xi \text{ and } \eta \text{ belong to the same cluster } C \text{ in } \mathfrak{M}' \text{ and } \xi \equiv_C \eta. \end{aligned}$$

Let $\mathfrak{M}'' = \langle W', R'', V' \rangle$. $\mathfrak{M}''$ is certainly finite and partially ordered. We claim that, for all $A \in \Psi$ and all $\xi \in W'$,

$$\mathfrak{M}', \xi \models A \quad \text{if and only if} \quad \mathfrak{M}'', \xi \models A.$$

The only difficulty in the inductive proof occurs in the inductive step at the following point. Suppose the assertion holds for B and that $\Box B \in \Psi$. With C a proper cluster in $\mathfrak{M}'$ and $\xi \in C$, assume that $\mathfrak{M}', \xi \not\models \Box B$. The difficulty consists in showing that $\mathfrak{M}'', \xi \not\models \Box B$. To solve it, notice that $\mathfrak{M}', \gamma_C \not\models \Box B$. Since γ_C is virtually last in C, it is possible to find $w \in \gamma_C$ such that, for all x, if $w\,R\,x$ then $\gamma_C = [w] = [x]$ or $[x] \notin C$. By the Filtration Theorem, $\mathfrak{M}, w \not\models \Box B$. Hence, for some x such that $w\,R\,x$, $\mathfrak{M}, x \not\models B$. By the Filtration Theorem again, $\mathfrak{M}', [x] \not\models B$. Hence, by the inductive hypothesis, $\mathfrak{M}'', [x] \not\models B$. Whether $[x] = \gamma_C$ or $[x] \notin C$, we have $\gamma_C\, R''\, [x]$. Since γ_C is $\leqslant_C$-last in C, $\xi\, R''\, \gamma_C$. Thus $\xi\, R''\, [x]$, and the desired conclusion follows: $\mathfrak{M}'', \xi \not\models \Box B$. □

Corollary 2.3.3. *The logic* **S4Grz** *is determined by the class of all finite reflexive trees.*

The proof is omitted; cf. Corollary 2.2.3.

Corollary 2.3.4. *The logic* **S4.3Grz** *is determined by the class of all finite linear orderings.*

Proof. Consistency is clear. The orderings $\leqslant_C$ in the proof of Theorem 2.3.2 are linear, so completeness is automatic. □

Corollary 2.3.5. *The logic* **S4.3Grz** *is determined by the single frame* $\langle \mathbb{N}, \geqslant \rangle$.

Proof. See Corollary 2.2.5. □

2.3.2 The schema Dum

We now turn to the study of the other new schema, Dum. First we observe that the proof of Lemma 2.3.1 can easily be turned into a proof of the following lemma:

Lemma 2.3.6. *Suppose L is a normal extension of* **S4Dum** *and that Ψ, $\mathfrak{M}$, and $\mathfrak{M}'$ are as in Lemma 2.3.1. Then every proper non-final cluster in $\mathfrak{M}'$ has a virtually last element.*

We shall also need the following fact.

Lemma 2.3.7. *Suppose L is a normal extension of* **S4Dum**. *Let Ψ be a logically finite formula set closed under subformulas and modalities. Suppose $\mathfrak{M} = \langle W, R, V \rangle$ is a point generated submodel of the canonical model for L, and suppose $\mathfrak{M}' = \langle W', R', V' \rangle$ is a Lemmon filtration of $\mathfrak{M}$ through Ψ. Then a final cluster in $\mathfrak{M}'$ is proper only if it is last in $\mathfrak{M}'$.*

Proof. Suppose C is a final cluster in $\mathfrak{M}'$. Assume that C is not last in $\mathfrak{M}'$. Then there exists some element $\zeta \in W'$ neither in C nor preceding C (nor, of course, succeeding C). Let ξ be any element in C. Assume, for any $A \in \Psi$, that

(1) $\mathfrak{M}', \xi \models A$.

We know (see the proof of Lemma 2.3.1 that we can find a Boolean combination B of formulas in Ψ such that, for each $\eta \in W'$,

$$\mathfrak{M}', \eta \models B \quad \text{if and only if} \quad \eta \in C.$$

By Theorem 1.7.7, for all $y \in W$,

(2) $\mathfrak{M}, y \models B$ if and only if $[y] \in C$.

Suppose t generates $\mathfrak{M}$. Take any $w \in \zeta$ and any $x \in \xi$. Clearly $t\,R\,w$ and $t\,R\,x$.

Since ζ does not precede C, it follows that, for every z such that $w\,R\,z$, $\mathfrak{M}, z \not\models \Diamond B$ and, hence, $\mathfrak{M}, z \not\models \Diamond B \to (B \wedge \neg A)$. Denote this formula by φ. Thus $\mathfrak{M}, w \models \Box\varphi$ and

(3) $\mathfrak{M}, t \models \Diamond\Box\varphi$.

Furthermore, if $[t] \in C$ then, because C is final, also $\zeta \in C$, which is absurd. Hence, by (2), $\mathfrak{M}, t \not\models B$. But also $\mathfrak{M}, t \models \Diamond B$. Hence

(4) $\mathfrak{M}, t \not\models \varphi$.

Since Dum is true in $\mathfrak{M}$, (3) and (4) yield

$$\mathfrak{M}, t \not\models \Box(\Box(\varphi \to \Box\varphi) \to \varphi),$$

so there exists some w such that $t\,R\,w$ and

(5) $\mathfrak{M}, w \models \Box(\varphi \to \Box\varphi)$;
(6) $\mathfrak{M}, w \models \Diamond B$; $\quad\quad \mathfrak{M}, w \not\models B \wedge \neg A$.

By (6), there is some z such that $w\,R\,z$ and $M\,z \models B$. It follows from (2) that $[z] \in C$.

Suppose one could find s such that $z\,R\,s$ and $\mathfrak{M}, x \not\models A$. Then $\mathfrak{M}, s \models \varphi$ (for C is final and hence $z\,R\,s$ implies that $[s] \in C$). Hence, by (5), $\mathfrak{M}, s \models \Box\varphi$, and consequently, $\mathfrak{M}, s \models \Box\neg A$. By hypothesis, Ψ is closed under modalities, so $\Box\neg A \in \Psi$. So, by the Filtration Theorem, $\mathfrak{M}', [s] \models \Box\neg A$. But then $\mathfrak{M}', \xi \models \neg A$, which is in contradiction with (1).

Conclusion: for every s, if $z\,R\,s$ then $\mathfrak{M}, s \models A$; thus $\mathfrak{M}, z \models \Box A$. Again, $\Box A \in \Psi$, so by the Filtration Theorem, $\mathfrak{M}', [z] \models \Box A$. Since $[z] \in C$, if follows that

(7) for any $\eta \in C$, $\quad \mathfrak{M}', \eta \models A$.

We have shown that (1) implies (7). Consequently, $C = \{\xi\}$. $\Box$

The notion of a kite was defined in Section 2.2. The reflexive version of this notion is now needed.

Theorem 2.3.8. *The logic* **S4Dum** *is determined by the class of all reflexive kites and all finite reflexive trees.*

Proof. Consistency is clear. Completeness is shown as in the case of Theorem 2.3.2, using Lemmata 2.3.6 and 2.3.7. It is important to notice that **S4Dum** has but finitely many distinct modalities (see Chapter 1, Theorem 1.6.3 and the remarks following it). $\Box$

Theorem 2.3.9. *The logic* **S4.3Dum** *is determined by the class of linear orderings of type* $\leqslant \omega$*, as well as by the single frame* $\langle \mathbb{N}, \leqslant \rangle$.

Cf. Corollaries 2.3.4 and 2.3.5.

It is easy to see that **S4.2Dum** is determined by the class of all reflexive kites, and **S4.2Grz** by the class of all finite reflexive kites. (A finite reflexive kite is nothing else but a finite convergent partially ordered frame!)

2.3.3 Related schemata

Analogously to what we did in Section 2.2, we may ask for a schema which, added to **S4Dum**, yields **S4Grz**. One such schema is

$\mathsf{M}.\quad \Box\Diamond A \to \Diamond\Box A.$

This is easy to see if one uses a result of [Segerberg, 1968b]: *if* Ψ, $\mathfrak{M}$, *and* $\mathfrak{M}'$ *are as in Lemma 2.3.7, except that the condition on* L *is that* L *is a normal extension of* **S4M***, then every final cluster of* $\mathfrak{M}'$ *is simple.* Thus

$$\mathbf{S4Grz} \quad = \quad \mathbf{S4MDum}.$$

(There is a syntactic proof of this already in [Sobociński, 1964a].)

Consider the following variations of Grz and Dum (for comparison, we give Grz and Dum here as well):

$\mathsf{Grz}.$	$\Box(\Box(A \to \Box A) \to A) \to A.$	$\mathsf{Dum}.$	$\Box(\Box(A \to \Box A) \to A) \to (\Diamond\Box A \to A).$
$\mathsf{Grz}_1.$	$\Box(\Box(A \to \Box A) \to A) \to \Box A.$	$\mathsf{Dum}_1.$	$\Box(\Box(A \to \Box A) \to A) \to (\Diamond\Box A \to \Box A).$
$\mathsf{Grz}_2.$	$\Box(\Box(A \to \Box A) \to \Box A) \to A.$	$\mathsf{Dum}_2.$	$\Box(\Box(A \to \Box A) \to \Box A) \to (\Diamond\Box A \to A).$
$\mathsf{Grz}_3.$	$\Box(\Box(A \to \Box A) \to \Box A) \to \Box A.$	$\mathsf{Dum}_3.$	$\Box(\Box(A \to \Box A) \to \Box A) \to (\Diamond\Box A \to \Box A).$

Given our completeness theorems for **S4Grz** and **S4Dum**, which take care of the most difficult parts, one can show that Grz, Grz_1, Grz_2, and Grz_3 are deductively equivalent in **S4**, as are Dum, Dum_1, Dum_2, and Dum_3. The question whether **S4Dum** = $\mathbf{S4Dum_3}$ is referred to in [Sobociński, 1970a] as an "unsolved problem of long standing." Thus, that problem is now solved in the positive. (To be certain, it remains to find a syntactic proof, but that task is left for somebody else.)

A schema due to Grzegorczyk (see [Grzegorczyk, 1967]) which has received some attention is

$$\Box(\Box(A \to \Box B) \to \Box B) \ \wedge \ \Box(\Box(\neg A \to \Box B) \to \Box B) \ \longrightarrow \ \Box B.$$

It is easy to find a syntactic proof that in **S4** this schema implies Grz. Using our completeness theorem for **S4Grz**, one sees that the schema is valid in **S4Grz**. Thus Grzegorczyk's schema is equivalent to Grz in **S4**. This settles another question of Sobociński [Sobociński, 1970b, p. 357].

2.4 Indices

In this dissertation, $\mathbb{N}$ denotes the set of natural numbers $0, 1, 2, \ldots$. Let $\mathbb{N}^*$ denote the set of non-positive integer numbers, and $\mathbb{Z}$ the set of all integers, negative as well as non-negative. Particularly in connection with indices, to be defined presently, we shall employ the identification the identification $m = \{0, \ldots, m-1\}$, for natural numbers m.

2.4.1 Definition of indices

Let T be either $\mathbb{Z}$ or $\mathbb{N}$ or $\mathbb{N}^*$ or a natural number m. We say that a sequence $i = \langle i_t \rangle_{t \in T}$ is an *index* if every i_t is either a natural number or ω. T is called the *index set*. For $t \in T$, i_t is the *coordinate* of t. If there is some c such that $i_t = c$, for all $t \in T$, then we sometimes write $i = c^T$. In particular, if $T = 1$, we may write $i = c$. When such simplifications are made in the notation it must be borne in mind that indices are essentially vectors and that, for example, 1^T is in general distinct from 1.

By the *length* of an index we understand the cardinality of its index set. An index is said to be *finite* if its length is finite and every coordinate is finite. Thus a finite index is simply an m-tuple of natural numbers, for some $m \in \mathbb{N}$.

We define the following binary relation $\precsim$ in the set of indices. Let $i = \langle i_t \rangle_{t \in T}$ and $j = \langle j_u \rangle_{u \in U}$ be indices. then $i \precsim j$ shall hold if there is a function $f : T \to U$ such that, for all $t, t' \in T$ and $u \in U$,

i. if $t < t'$ then $f(t) < f(t')$;
ii. if T has a greatest element t'' then $f(t'')$ is the greatest element of U;
iii. $i_t \leqslant j_{f(t)}$;
iv. if $i_t = 0$ then $j_{f(t)} = 0$;
v. if $j_u = 0$, then either there are no $t_0 \in T$ and $u_0 \in U$ such that $u_0 < u$ and $f(t_0) = u_0$, or there is some $t_1 \in T$ such that $f(t_1) = u$, and $f(t_1 - 1) = u - 1$.

Such a function f is called an *index function* from i to j. It is clear that $\precsim$ is reflexive and transitive, but not in general antisymmetric. However, the restriction of $\precsim$ to the set of finite indices is antisymmetric:

Lemma 2.4.1. *If i and j are finite indices, then $i \precsim j$ and $j \precsim i$ implies that $i = j$.*

Proof. As i and j are finite, they have finite index sets, say m and n, respectively. Suppose f and g are index functions from i to j, and from j to i, respectively. Since f and g are both injective, $m \leqslant n$ and $n \leqslant m$, so $m = n$. For all $t, t' \in m$,

$$t < t' \implies f(t) < f(t') \implies g(f(t)) < g(f(t')),$$

so both f and g must be the identity transformation on m. For every $t \in m$,

$$i_t \leqslant j_{f(t)} = j_t \quad \text{and} \quad j_t \leqslant i_{g(t)} = i_t,$$

so, for every $t \in m$, $i_t = j_t$. $\Box$

2.4.2 Index of a frame

Our motivation for introducing indices is that they provide a convenient means for talking about an important class of generated connected transitive frames. Suppose $F = \langle W, R\rangle$ is a frame with these properties. Let $\sim$ have the usual meaning (see Section 2.1). Suppose F is such that there can be found a set T of integers such that T is either $\mathbb{Z}$ or $\mathbb{N}$ or $\mathbb{N}^*$ or m, for some $m \in \mathbb{N}$, and such that T enumerates the set $W/{\sim}$ of clusters in F. In other words, we assume that there is a bijection $z\colon T \to W/{\sim}$ such that, for all $t, t' \in T$,

$$t < t' \quad \text{if and only if} \quad z(t) \text{ precedes } z(t').$$

Let $i(z(t))$ be the number of reflexive elements in $z(t)$; if $z(t)$ is infinite then $i(z(t)) = \omega$. Then the sequence $\langle i(z(t))\rangle_{t\in T}$ is an index. We shall refer to this index as *the index of* F, and denote it by $\mathsf{index}(F)$. Thus the index brings out the structure of F; two countable frames have the same index only if they are isomorphic, and, if a frame has an index then any isomorphic frame has the same index. A frame that has an index is called an *index frame*. Of course, only a generated connected transitive frame can be an index frame. If F is an index frame, then the logic $\mathsf{Logic}(F)$ determined by F is called an *index logic*. For example, a logic determined by a frame with index i may be said to be *determined by* i. We have already met with several index logics. For example, the following logics have the following indices:

K4.3W	0^{ω^*}	**S4.3Grz**	1^{ω^*}
D4.3Z	0^{ω}	**S4.3Dum**	1^{ω}

(Here ω^* is the order type of $\mathbb{N}^*$.) Note that an index logic need not have a unique index. For example, **S4.3Dum** has also index 1^{ζ}, where ζ is the order type of $\mathbb{Z}$.

It may well be asked what we lose by not permitting higher cardinalities than countable in the indices. The answer is that we loose nothing; the sense in which this has to be understood is made clear by the following theorem.

Theorem 2.4.2. *Suppose F is a generated connected transitive frame containing a non-denumerable cluster C. Let F° be the frame that is just like F except for having instead of C a denumerable cluster C° that is a subset of C. Then F and F° determine the same logic.*

Proof. We must prove that $\mathsf{Logic}(F) = \mathsf{Logic}(F^\circ)$. It may be assumed, with no loss of generality, that $C^\circ = \mathbb{N}$.

First assume that $A \notin \mathsf{Logic}(F^\circ)$. Let V be a valuation in F° such that A is rejected by a model M defined on F° by V. Let f be the function defined on the domain of F such that

$$f(x) \;=\; \begin{cases} x, & \text{if } x \notin C \text{ or } x \in \mathbb{N}, \\ 0, & \text{if } x \in C \setminus \mathbb{N}. \end{cases}$$

Define a valuation $V'(p_n) = \{x \mid f(x) \in V(p_n)\}$, for every n. Let M' be the model defined on F by V'. Then f is a p-morphism from M' to M that is reliable on every propositional letter. By the P-morphism Theorem, M' and M are equivalent, so A fails somewhere in M'. Thus, $A \notin \mathsf{Logic}(F)$.

Next assume that $A \notin \mathsf{Logic}(F)$. Let V be a valuation in F such that A is falsified by the model M defined on F by V. We may assume without loss of generality that $V(p_n) = \varnothing$ for every n such that p_n does not occur in A. Let Ψ be the set of subformulas of A (finite!), and $\equiv_\Psi$ be the same equivalence relation, restricted to C, as described in Section 1.7:

$$x \equiv_\Psi y \quad \text{iff, for every } B \in \Psi,\ \ M, x \models B \text{ iff } M, y \models B.$$

Let $\xi_0, \ldots, \xi_m$ be the equivalence classes in C under $\equiv_\Psi$. At least one of them, say ξ_m, must be non-denumerable. Let g be a bijective map from a denumerable subset D of ξ_m to $\mathbb{N}$. We then define this function for elements x of the domain of F:

$$f(x) \;=\; \begin{cases} x, & \text{if } x \notin C; \\ i, & \text{if } x \in \xi_i \text{ and } i < n, \\ & \text{or } x \in \xi_n \setminus D \text{ and } i = n, \\ g(x) + n + 1, & \text{if } x \in D. \end{cases}$$

Define a valuation $V^\circ(p_n) = \{x \mid f(x) \in V(p_n)\}$, for every n. This definition is easily seen to be meaningful. Let M° be the model defined on F° by V°. Then f is a p-morphism from M to M° that is reliable everywhere. Hence A fails somewhere in M° and thus $A \notin \mathsf{Logic}(F^\circ)$. $\square$

2.4.3 The Index Theorem

Suppose $i = \langle i_t \rangle_{t \in T}$ is an index. Define

$$W \;=\; \bigcup\{\, \langle t,n\rangle \mid t \in T \text{ and } 0 \leqslant n \leqslant i_t \,\};$$
$$\langle t,n\rangle \, R \, \langle t',n'\rangle \quad \text{iff} \quad \text{either } t < t', \text{ or } (t = t' \text{ and } i_t > 0).$$

Then $F = \langle W, R\rangle$ is a frame, and it is easy to see that $\mathsf{index}(F) = i$. F is called the *special frame* of index i. It is often convenient to identify a given index frame and the special frame of the same index. According to the preceding theorem, as long as we are interested only in what logics the frames determine, such an identification is permissible.

Suppose F and G are special frames of indices i and j, respectively. Let f be an index function from i to j. By the *index morphism from F to G induced by f* we mean the function $\overline{f}$ such that, for every element $\langle t,n\rangle$ in F,

$$\overline{f}(\langle t,n\rangle) \;=\; \langle f(t), n\rangle.$$

That this definition is meaningful is guaranteed by clause (iii) in the definition of $\precsim$:

(a) x precedes y in F iff $\overline{f}(x)$ precedes $\overline{f}(y)$ in G;
(b) F has a last cluster iff G has a last cluster. If $x/{\sim}$ is last in F, then $\overline{f}(x)/{\sim}$ is last in G.
(c) $x/{\sim}$ is degenerate in F iff $\overline{f}(x)/{\sim}$ is degenerate in G;
(d) If D is a degenerate cluster in G, then there exists some y in F such that $\overline{f}(y) \in D$. Let $x/{\sim}$ be the cluster immediately preceding $y/{\sim}$, if it exists. Then $\overline{f}(x)/{\sim}$ immediately precedes D.

Theorem 2.4.3 (The Index Theorem).
If F and G are index frames and $\mathsf{index}(F) \precsim \mathsf{index}(G)$, *then* $\mathsf{Logic}(F) \supseteq \mathsf{Logic}(G)$.

Proof. Suppose A is a formula such that $A \notin \mathsf{Logic}(F)$. Then there is a valuation V in F that falsifies A. In order to prove the theorem it will be enough to define a valuation in G that falsifies A.

We may assume with impunity that $F = \langle W, R\rangle$ and $G = \langle W', R'\rangle$ are special. Let f be an index function from $\mathsf{index}(F)$ to $\mathsf{index}(G)$, and let $\overline{f}$ be the index morphism induced by f. For every cluster C in F let γ_C be an arbitrary fixed element. We define a function $g\colon W' \to W$ as follows:

$$g(w) \;=\; \begin{cases} \overline{f}^{-1}(w), & \text{if } w \in \overline{f}[W]; \\ \gamma_C, & \text{if } w \notin \overline{f}[W] \text{ and } C \text{ is the cluster in } F \text{ such that } w/{\sim} \text{ does not succeed } f(\gamma_C)/{\sim}. \end{cases}$$

It follows from clauses (iii) and (v) of the definition of $\precsim$ that in order to show that g is well-defined it will suffice to show that for every reflexive $w \in W' \setminus \overline{f}[W]$ there is some $x \in \overline{f}[W]$ such that $w \, R' \, x$. There are two cases.

(1) The index set of $\mathsf{index}(F)$ has a last element. Then, by clause (ii) of the definition of $\precsim$, so does the index set of $\mathsf{index}(G)$. This is to say that FF has a last cluster C, and G has a last cluster D, and $\overline{f}[C] \subseteq D$. If any $w \in W' \setminus \overline{f}[W]$ does not precede D, then $w \in D \setminus \overline{f}[W]$, whence D is not degenerate. Hence $w \, R' \, f(x)$, for any $x \in C$.

(2) The index set of $\mathsf{index}(F)$ does not have a last element. Then F has no last cluster. Take any $w \in W' \setminus \overline{f}[W]$. If no element in G precedes w, then for any $z \in W$, $w \, R' \, f(z)$ (since w is reflexive). So assume some elements precede w in G. Of the clusters D such that D precedes $w/{\sim}$ and $D \cap \overline{f}[W] \neq \varnothing$ there will be a last one, because the set of clusters in any index frame is discrete. Let D_0 be that cluster. Take any $x \in W$ such that $\overline{f}(x) \in D$. Since F has no last cluster, there exists a cluster succeeding $x/{\sim}$; call in C. Take any $y \in C$. Clearly $f(y)/{\sim}$ cannot precede $w/{\sim}$. If $w/{\sim}$ precedes $f(y)/{\sim}$, we are through. Suppose $w/{\sim}$ and $f(y)/{\sim}$ are the same cluster. Since w is reflexive, $w/{\sim}$ is not degenerate, so $w \, R' \, f(y)$.

We define, for all n, $V'(p_n) = \{w \in W' \mid g(w) \in V(p_n)\}$. Let M be the model defined on F by V. By M' we shall understand the model generated from $\langle W', R', V'\rangle$ by some element $\overline{f}(x)$ such that $x/{\sim}$ is the first cluster in F, if a first cluster exists; otherwise $M = \langle W', R', V'\rangle$. Note that this definition of M' is meaningful. We claim that g is a p-morphism from M' to M, reliable everywhere. This claim implies that the formula A is rejected by some model on G, so to prove the theorem it will be enough to prove the claim.

We make the following observations, (A)–(C), where w is a parameter taking values in the domain of M'.

(A) If $w \notin \overline{f}[W]$ then $w/{\sim}$ is nondegenerate.

Proof: If $w/{\sim}$ is degenerate, then it follows from clause (v) in the definition of $\precsim$ that there exists some $x \in W$ such that $\overline{f}(x) = w$.

(B) $w/{\sim}$ is degenerate if and only if $g(w)/{\sim}$ is degenerate.

Proof: Suppose $w/{\sim}$ is degenerate. By (A), there exists some $x \in W$ such that $w = f(x)$. By clause (iii) of the definition of $\precsim$, $g(w)/{\sim}$ is degenerate. Conversely, suppose $g(w)/{\sim}$ is degenerate. Then, by clause (iv) of the definition of $\precsim$, $\overline{f}(g(w))/{\sim}$ is degenerate.

If $w \in \overline{f}[W]$ then $\overline{f}(g(w)) = w$, so $w/{\sim}$ is degenerate.

If $w \notin \overline{f}[W]$ then $w/{\sim}$ does not succeed $\overline{f}(g(w))/{\sim}$. By (A), $w/{\sim}$ is nondegenerate, so $w/{\sim}$ and $\overline{f}(g(w))/{\sim}$ are distinct. Hence $w/{\sim}$ precedes $\overline{f}(g(w))$. By clause (v) of the definition of $\precsim$, then, there is a cluster C immediately preceding $g(w)/{\sim}$ and $\overline{f}(\gamma_C)/{\sim}$ immediately precedes $\overline{f}(g(w))/{\sim}$. Since $w \in W'$, the model M' cannot be generated by $\overline{f}(g(w))$, and therefore $g(w)/{\sim}$ cannot be first in F. By clause (v) of the definition of $\precsim$, then, if C is the cluster immediately preceding $g(w)/{\sim}$, then $\overline{f}(\gamma_C)/{\sim}$ immediately precedes $\overline{f}(g(w))/{\sim}$. Hence $w/{\sim}$ does not succeed $\overline{f}(\gamma_C)/{\sim}$. But then $g(w) = \gamma_C$, which is absurd.

(C) If $w \notin \overline{f}[W]$ then $g(w)/{\sim}$ is nondegenerate.

Proof: Follows from (A) and (B).

We shall now check the three conditions in the definition of p-morphism (the fourth condition is automatically satisfied).

(i) That g is onto is clear.

(ii) Assume that $w\,R'\,x$. We must prove that $g(w)\,R\,g(x)$. There are two cases.

  (1) $w \sim x$. Then $w/{\sim}$ is nondegenerate. Then, by (B), $g(w)/{\sim}$ is nondegenerate. As, clearly, $g(w)$ and $g(x)$ belong to the same cluster, it follows that $g(w)\,R\,g(x)$.

  (2) w precedes x. It follows from the definition of g that $g(w)$ cannot succeed $g(x)$. Suppose $g(w) = g(x)$ and $g(w)$ is irreflexive. Then, by (C), $w, x \in \overline{f}[W]$. But $\overline{f}$ is injective. Hence $w = x$, which is impossible. It follows that $g(w)\,R\,g(x)$.

(iii) Assume that $g(w)\,R\,g(x)$. We must prove that $w\,R'\,y$, for some y such that $g(x) = g(y)$.

If $g(w)$ and $g(x)$ belong to different clusters, it is clear that w precedes x, hence that $w\,R'\,x$.

Assume therefore that $g(w) \sim g(x)$ and that x does not succeed w. If $w = x$ then $w/{\sim}$ is nondegenerate, and hence $w\,R'\,x$. The real difficulty lies in the remaining possibility: that x precedes w. Then it is impossible that $x \in \overline{f}[W]$, so $g(x) = \gamma_{g(w)/\sim}$. If $w \notin \overline{f}[W]$ then y can be taken as $\overline{f}(\gamma_{g(w)/\sim})$. If $w \in \overline{f}[W]$ then $w \sim \overline{f}(\gamma_{g(w)/\sim})$ and we must only make certain that $w/{\sim}$ is nondegenerate. But this certainty is afforded by (A).

This conclues the proof of Theorem 2.4.3. □

Corollary 2.4.4. *If $\mathcal{C}$ and $\mathcal{D}$ are classes of index frames such that for every $F \in \mathcal{C}$ there exists some $G \in \mathcal{D}$ such that* $\mathsf{index}(F) \precsim \mathsf{index}(G)$, *then* $\mathsf{Logic}(\mathcal{C}) \supseteq \mathsf{Logic}(\mathcal{D})$.

Proof. By Theorem 1.3.3, $\mathsf{Logic}(\mathcal{C}) = \bigcap\{\mathsf{Logic}(F) \mid F \in \mathcal{C}\}$ and similarly for $\mathcal{D}$. □

Corollary 2.4.5. *Suppose that $\mathcal{C}$ is a class of index frames containing an index frame F such that, for all $G \in \mathcal{C}$,* $\mathsf{index}(G) \precsim \mathsf{index}(F)$. *Then* $\mathsf{Logic}(\mathcal{C}) = \mathsf{Logic}(F)$.

Corollary 2.4.6. *Suppose that $\mathcal{C}$ and $\mathcal{D}$ are classes of index frames such that $\mathcal{D}$ is a subclass of $\mathcal{C}$ and for every $F \in \mathcal{C}$ there exists some $G \in \mathcal{D}$ such that* $\mathsf{index}(F) \precsim \mathsf{index}(G)$. *Then* $\mathsf{Logic}(\mathcal{C}) = \mathsf{Logic}(\mathcal{D})$.

2.5 Normal extensions of K45

What extensions of the logic **S5** are there? Certainly, the Trivial System (which we shall abbreviate **Triv**) is one: The system obtained by adding the schema

N'. $\quad A \to \Box A$

to an axiomatization of T. (It is easy to see that the logic **Triv** is determined by the class of frames that consist of one single reflexive element. Thus **Triv** is of index 1.) Furthermore, for each positive integer n, the normal extension of **S5** by Alt_n is determined by the index frame n and is a proper extension of **S5**. (See Theorem 1.5.4.) That each $\mathbf{S5Alt}_n$ includes $\mathbf{S5Alt}_{n+1}$ follows from the Index Theorem, and that the inclusion is proper is easy to see. Thus there is a denumerable family of strictly descending logics which all properly contain **S5**:

$$\mathbf{S5Alt}_1 \supset \mathbf{S5Alt}_2 \supset \ldots \supset \mathbf{S5Alt}_n \supset \ldots \supset \mathbf{S5}.$$

($\mathbf{S5Alt}_1$ is, of course, nothing but the Trivial System.) Does this sequence comprise all the normal extensions of **S5**? A positive answer was given in [Scroggs, 1951]. This is one of the classical results in modal logic, and we shall refer to it under the name of Scroggs' First Theorem. In this section we shall reconstruct Scroggs' proof and show how his result can be improved: the normal extensions of **K45** can be handled much the same way Scroggs handled the normal extensions of S5.

Theorem 2.5.1. *Every normal extension of* **K45** *has the finite model property.*

Proof. Let L be a normal logic such that $\mathbf{K45} \subseteq L$. Suppose a certain formula A is not derivable in L. Then A is false somewhere in the canonical modal $\mathfrak{M}_L$ for L, say at t. Let $\mathfrak{M} = \langle W, R, V\rangle$ be the submodel of $\mathfrak{M}_L$ generated by t. By the generation theorem, A fails in $\mathfrak{M}$ at t. We shall show how $\mathfrak{M}$ can be used to define a model $\mathfrak{M}'$ for L that rejects A and is finite.

Since $\mathbf{K45} \subseteq L$, the model $\mathfrak{M}$ consists of no more than two clusters, and if there are two clusters then the first, but not the last, is degenerate. (See [Segerberg, 1968a].) Thus there are three possibilities:

(i) $\mathfrak{M}$ consists of one irreflexive element – t;
(ii) $\mathfrak{M}$ consists of one nondegenerate cluster;
(iii) $\mathfrak{M}$ consists of one irreflexive element – t – and a nondegenerate cluster.

In case (iii) we shall assume, with no loss of generality, that A is true throughout the last cluster (if A were actually false somewhere in the last cluster, we could find a new generated submodel of $\mathfrak{M}_L$ falling under (ii).) Let Ψ be the smallest set containing A and closed under subformulas and modalities. We give a definition of $\mathfrak{M}'$ by considering each of the three cases separately.

(i) $\mathfrak{M}'$ shall be $\mathfrak{M}$, simply.
(ii) $\mathfrak{M}'$ shall be the filtration of $\mathfrak{M}$ through Ψ such that every propositional letter not in Ψ is false everywhere in $\mathfrak{M}'$. (It is readily seen that $\mathfrak{M}'$ is indeed uniquely defined.)
(iii) This is the most complicated case. Let write $Y = W \setminus \{t\}$. We define $\mathfrak{M}' = \langle W', R', V'\rangle$, where

$$\begin{aligned} W' &= \{\langle 0,[t]\rangle\} \cup \{\langle 1,[w]\rangle \mid w \in Y\}; \\ \langle i,[w]\rangle\, R\, \langle j,[x]\rangle \quad &\text{iff} \quad i<j, \text{ or } i=j \text{ and } w\,R\,x; \\ V'(p_n) &= \{\langle i,[w]\rangle \mid p_n \in \Psi \text{ and } w \in V(p_n)\}. \end{aligned}$$

(Here the square brackets have the same use as in Section 1.7.)

In either of the three cases $\mathfrak{M}'$ will be finite (the logic **K45** contains only finitely many non-equivalent modalities; see also Theorem 1.6.3.) In each case there is an obvious p-morphism from $\mathfrak{M}$ to $\mathfrak{M}'$ which is reliable on the set of propositional letters in Ψ. Consequently, $\mathfrak{M}$ and $\mathfrak{M}'$ are equivalent modulo the set Ψ (and even modulo the set Σ of formulas with propositional letters in Ψ), so A fails in $\mathfrak{M}'$. The only remaining task then is to prove that $\mathfrak{M}'$ is a model for L.

Suppose that B a formula that is rejected by $\mathfrak{M}'$. Let B^* be the result of substituting $\perp$ for every propositional letter in B that is not in Ψ. Then, because the valuation of $\mathfrak{M}'$ has been defined to make propositional letters not in Ψ false throughout $\mathfrak{M}'$, B^* is also rejected by $\mathfrak{M}'$. (This informal argument is easily turned into an inductive formal proof.) Since $\mathfrak{M}$ and $\mathfrak{M}'$ are equivalent modulo Σ and $B^* \in \Sigma$, B^* is rejected by M as well. Hence B^* is not a theorem of L. Nor is B, then, for B^* is a substitution instance of B, and L is closed under substitution. □

Remark. The model $\mathfrak{M}'$ is distinguishable. In case (iii) this follows because of the special proviso.

As has been remarked before, an important unsolved problem about modal logics is whether all are determined. The preceding theorem, extending Scroggs' First Theorem, gives a partial, if extremely humble, answer to that question. For, considering Theorem 1.3.7, we have shown that every non-theorem of a normal extension of **K45** has a counter-frame of finite index: either 0, or k, or $\langle 0, n\rangle$ (for some positive integers k, n). This means that every normal extension of **K45** is determined by some class of such frames – just take one counter-frame for each non-theorem. In fact, in view of the Index Theorem and the proof of Theorem 2.5.1, it is clear that every such logic is determined by a class of at most three index frames, with the possible candidates to be sought among 0, k, and $\langle 0, n\rangle$ (for some k, n such that $0 < k, n \leqslant \omega$); if both a frame k and a frame $\langle 0, n\rangle$ belong to that class, then one may assume $k > n$. Consequently, the only index logics at least as strong as **K45** are **D45**, **D45Alt**$_n$, **S5**, **S5Alt**$_n$, and **Abs**, where n ranges over the set of positive integers, and **Abs** is the Absurd System **EQ** defined in Section 1.4. (As to the name, think of **EQ** as a deontic logic!) The systems mentioned are at the same time the only monolithic logics at least as strong as **K45** (see Section 1.3).

Our new-won insights enable us to classify the entire family of normal extensions of **K45**. The chart given in Figure 2.1 brings out its structure. As usual in such charts, if two systems are joined by one straight line segment then one is properly included in the other; in our charts the lower system will in general be the weaker In the chart are indicated the indices of the frames of classes that determine some of the systems. For example, "$0, \omega$" by the point representing **K4B** indicates that **K4B** is determined by the class whose elements are the index frames 0 and ω. Notice that there are no intercalate (??????????) systems except where the lines are dotted. In particular, there are no logics between the two main "slices" of logics, for example, between **K45** and **D45**, or between **K4B** and **S5**.

Figure

Figure 2.1: Figure.

By Theorem 1.3.3, there are several intersection results forthcoming. In fact, every normal extension of **K45** is the intersection of at most three index logics. For example,

$$\begin{array}{ccccc} \mathbf{K45} & = & \mathbf{D45} & \cap & \mathbf{Abs}, \\ \mathbf{K4B} & = & \mathbf{S5} & \cap & \mathbf{Abs}. \end{array}$$

Clearly every index logic stronger than **K45** is decidable. But every intersection of finitely many decidable logics is itself decidable. Hence there is this general result:

Theorem 2.5.2. *Every normal extension of the logic* **K45** *is decidable.*

2.6 Normal extensions of K4

Having given, in Section 2.5, an exhaustive account of all normal extensions of **K45**, we may ask whether our analysis can be extended to other classes of logics. In this section we shall present some partial results along this line. Unfortunately, our efforts will not lead to anything near the complete success of the preceding section. Much future work is called for in this area.

We begin by proving the following basic result.

Theorem 2.6.1. *Suppose L is a classical (not necessarily normal) logic such that, for every natural number n, there are only finitely many nonequivalent propositional functions in n variables. Then L has the finite model property.*

Proof. Suppose that A is any non-theorem of L. Then A fails in the canonical neighborhood model $\mathfrak{M} = \langle W_L, \mathcal{N}_L, V_L\rangle$ for L. Let Ψ be the set of subformulas of A, and let Ψ' be the closure of Ψ under propositional functions (that is, Ψ' will contain all formulas with propositional letters among those occurring in A). It follows from the assumption of the theorem that Ψ' is logically finite. Let $\mathfrak{M}' = \langle W', \mathcal{N}', V\rangle$ be any neighborhood filtration of $\mathfrak{M}_L$ through Ψ' such that, for every p_n that does not occur in A, $V'(p_n) = \varnothing$. Clearly, for every formula $B \in \Psi'$ and every $w \in W_L$,

$$\mathfrak{M}_L, w \models B \quad \text{if and only if} \quad \mathfrak{M}', [w] \models B.$$

In particular, A fails in $\mathfrak{M}'$. Suppose now that C is a theorem of L that fails in $\mathfrak{M}'$. Let C^* be the result of substituting $\perp$ for every propositional letter in C which does not occur in A. Then C^* fails in $\mathfrak{M}'$ and hence, since $C^* \in \Psi'$, C^* fails in $\mathfrak{M}_L$. As C^* is a substitution instance of C, this is impossible. Thus $\mathfrak{M}'$ is a model for L. That $\mathfrak{M}'$ is finite is plain. $\square$

This subcase is of special interest in this section:

Theorem 2.6.2. *Suppose L is a normal logic such that, for every natural number n, there are only finitely many nonequivalent propositional functions in n variables. Then L has the finite model property.*

Apart from Scroggs' First Theorem, the most famous result of the kind exemplified by Theorem 2.6.2 is Bull's Theorem, which states that every normal extension of **S4.3** has the f.m.p. (see [Bull, 1966]). It should be noted that our result does not generalize Bull's Theorem (although one may say that it does generalize Scroggs' First Theorem); as Makinson has shown, even **S4.3Grz** has infinitely many propositional functions in one variable already (see [Makinson, 1966b]). Given this limitation on our result, it would be interesting to know what its worth is. In other words; what interesting systems are there with only finitely many nonequivalent propositional functions for each number of variables? Each system $\mathbf{K4Alt}_n$ is of this sort, but that hardly counts as an interesting fact since each of those systems is finite. Leaving the general question posed unanswered, we shall devote the remainder of the section to analysing one nontrivial family of logics, which will turn out to belong to the delineated category.

2.6.1 Completeness of $\mathbf{K4B}_n$

We recursively define the following family of schemata ($n \geqslant 1$):

$$\Gamma(A_1, \ldots, A_n). \qquad \Diamond(\Box A_1 \wedge \neg\Gamma(A_2, \ldots, A_n)) \to A_1.$$

Notice that $\Gamma(A_1)$ is the schema $\Diamond\Box A_1 \to A_1$, which is nothing else but the schema B. For this reason let B_n denote the schema $\Gamma(A_1, \ldots, A_n)$. We shall presently give a completeness theorem for each $\mathbf{K4B}_n$.

Suppose F is a transitive frame. By a *tower* of clusters in F we mean a collection of clusters in F such that each cluster precedes or succeeds every other cluster. The tower is *on* C if C is a cluster in the tower preceding every other cluster in the tower. The *rank* of a cluster C, denoted by $\rho(C)$, is defined as the integer n (if it exists) such that no tower on C consists of $> n$ clusters, and some tower on C consists of exactly n clusters; if no such n exists, the rank is undefined. An element x has rank $\rho(x)$ equal to $\rho(x/{\sim})$. A frame F has rank n is no cluster in F has rank $> n$, and some cluster has rank n. Elements and frames lack ranks if these conditions are not met.

Theorem 2.6.3. *The schema B_n is valid in every transitive frame of rank $\leqslant n$.*

Proof. Suppose there is a transitive model $M = \langle W, R, V\rangle$ and formulas $A_1, \ldots, A_n$ such that $\Gamma(A_1, \ldots, A_n)$ is rejected by M. Then there will exist points $x_1, \ldots, x_{n+1}$ such that, if $1 \leqslant i \leqslant n$,

$$x_i \, R \, x_{i+1}, \qquad x_i \not\models A_i, \qquad x_{i+1} \models A_i, \qquad x_i \not\models \Gamma(A_1, \ldots, A_n).$$

(This claim is easily verified by induction on i.) Suppose there are i, j such that $1 \leqslant i < j \leqslant n+1$ and x_i and x_j belong to the same cluster C. Because R is transitive and $j \geqslant i+1$, we have $x_j \models \Box A_i$. If C were nondegenerate, then $x_j \, R \, x_i$ and hence $x_i \models A_i$, which is absurd. Hence C is degenerate, and thus $x_i = x_j$. But $x_i \, R \, x_j$, which now implies that x_i is reflexive, and that is impossible if C is degenerate.

The conclusion is that such i and j cannot be found. The set $\{x_1/{\sim}, \ldots, x_{n+1}/{\sim}\}$ therefore forms a tower of $n+1$ distinct clusters, so the frame of M cannot have rank $\leqslant n$. $\square$

Theorem 2.6.4. *If L is a normal extension of $\mathbf{K4B}_n$ then the canonical model $\mathfrak{M}_L$ for L has rank $\leqslant n$.*

Proof. Let $\{C_1, \ldots, C_{n+1}\}$ be a tower in $\mathfrak{M}_L$, C_i preceding C_{i+1} whenever $1 \leqslant i \leqslant n$. Then there exist formulas A_i such that, for $1 \leqslant i \leqslant n$,
– for every $w \in C_{i+1}$, we have $w \models \Box A_i$;
– for every $w \in C_i$, we have $w \not\models A_i$.

An inductive argument easily establishes that, for $1 \leqslant i \leqslant n$,

$$\text{for some } w \in C_i, \text{ we have } w \not\models \Gamma(A_i, \ldots, A_n).$$

In particular, there exists $w \in C_1$ such that $w \not\models \Gamma(A_1, \ldots, A_n)$. But in $\mathfrak{M}_L$, every instance of B_n is true everywhere. □

It is an easy corollary of the last two theorems that if L is a normal extension of $\mathbf{K4B}_n$, then the canonical model for L has rank equal to n. Hence we have two completeness results: the logic $\mathbf{K4B}_n$ is determined by the class of frames of rank $\leqslant n$, and also by the class of frames of rank n. Another result worth mentioning is that the logic $\mathbf{K4.3B}_n$ ($\mathbf{D4.3B}_n$, $\mathbf{S4.3B}_n$) is determined by the class of index frames of length n (where the last cluster is nondegenerate; where no cluster is degenerate). Also observe that

$$\begin{array}{rclcrcl}
\mathbf{K4} & = & \bigcap_n \mathbf{K4B}_n; & \quad & \mathbf{K4.3} & = & \bigcap_n \mathbf{K4.3B}_n; \\
\mathbf{D4} & = & \bigcap_n \mathbf{D4B}_n; & & \mathbf{D4.3} & = & \bigcap_n \mathbf{D4.3B}_n; \\
\mathbf{S4} & = & \bigcap_n \mathbf{S4B}_n; & & \mathbf{S4.3} & = & \bigcap_n \mathbf{S4.3B}_n.
\end{array}$$

We mention in passing an easier way of axiomatizing the 4.3-systems. Consider (for $n \geqslant 1$, cf. Alt_n):

C'_n. $\quad A_1 \lor \Box(A_1 \land \Box A_1 \to A_2) \lor \ldots \lor \Box(A_n \land \Box A_n \to A_{n+1})$.

C_n. $\quad A_1 \lor \Box(\Box A_1 \to A_2) \lor \ldots \lor \Box(\Box A_n \to A_{n+1})$.

Then

$$\begin{array}{rclcl}
\mathbf{K4.3B}_n & = & \mathbf{K4.3C}'_n; & & \\
\mathbf{D4.3B}_n & = & \mathbf{D4.3C}'_n; & & \\
\mathbf{S4.3B}_n & = & \mathbf{S4.3C}'_n & = & \mathbf{S4.3C}_n.
\end{array}$$

2.6.2 Hereditary f.m.p. of $\mathbf{K4B}_n$

Theorem 2.6.5. *For each $k \in \mathbb{N}$, there are only finitely many nonequivalent propositional functions in k variables in $\mathbf{K4B}_n$.*

Proof. For simplicity, let $\mathfrak{M} = \langle W, R, V \rangle$ be the canonical model for L. Let k be any fixed natural number. Let $\Pi = \{p_0, \ldots, p_{k-1}\}$ and let Ω be the set of all formulas with propositional letters in Π. If S is any subset of W, we shall let S/Π be the set of equivalence classes under $\equiv_\Pi$ of elements in S (see Section 1.7). The symbol $\sim$ shall have the usual meaning (see Section 2.1). We now define the following binary relation between generated submodels of $\mathfrak{M}$:

$\mathfrak{M}^x \simeq \mathfrak{M}^y$ iff (i) $\rho(x) = \rho(y)$; and
(ii) $(x/{\sim})/\Pi = (y/{\sim})/\Pi$; and
(iii) for each cluster E, if E immediately succeeds $x/{\sim}$,
then there exists a cluster F immediately succeeding $y/{\sim}$
such that $E \simeq F$ (???), and conversely.

Because of Theorem 2.6.4, this definition is meaningful. Clearly $\simeq$ is an equivalence relation. We now define the following relation between elements of W:

$$x \approx y \quad \text{iff} \quad x \equiv_\Pi y \ \text{ and } \ \mathfrak{M}^x \simeq \mathfrak{M}^y.$$

It is clear that also $\approx$ is an equivalence relation. Let $W/{\approx}$ be the set of equivalence classes $w/{\approx}$ of elements $w \in W$. We define a binary relation $R/{\approx}$ on $W/{\approx}$ and a function $V/{\approx}$ on Var:

$$(x/{\approx})\, R/{\approx}\, (y/{\approx}) \quad \text{iff} \quad \text{there are } x', y' \text{ such that } x \approx x',\ y \approx y', \text{ and } x' R y';$$
$$(V/{\approx})(p_n) \;=\; \{\, x/{\approx} \mid p_n \in \Pi \text{ and } x \in V(p_n) \,\}.$$

Let $\mathfrak{M}/{\approx} \;=\; \langle W/{\approx}, R/{\approx}, V/{\approx}\rangle$. Suppose for the moment that we have proved that

(A) $W/{\approx}$ is finite;
(B) there is a p-morphism from $\mathfrak{M}$ to $\mathfrak{M}/{\approx}$ reliable on every propositional letter in Ω.

It follows from (A) that there can be found a finite set $\Omega_0 \subseteq \Omega$ such that for every $A \in \Omega$ there is some $A_0 \in \Omega_0$ such that $A \leftrightarrow A_0$ is true in $\mathfrak{M}/{\approx}$. Now (B) implies that $\mathfrak{M}$ and $\mathfrak{M}/{\approx}$ are equivalent modulo Ω and $\mathfrak{M}$ is the canonical model for $\mathbf{K4B}_n$. Therefore Ω_0 is a finite base for Ω in $\mathbf{K4B}_n$. In other words, Ω is logically finite in $\mathbf{K4B}_n$, so by Theorem 1.6.1, there are at most finitely many nonequivalent propositional functions of k variables in $\mathbf{K4B}_n$. This is what we want to show, so it only remains to prove (A) and (B).

It is easy to see that (A) is established by proving that there are only finitely many equivalence classes under $\simeq$ in the class of generated submodels of $\mathfrak{M}$. By induction on the rank r, one can show that, for every $r \leqslant n$, there are only finite many nonequivalent generated submodels of rank r. What it all boils down to is that, for any cluster C, C/Π is finite because Π is finite.

As to (B), it is sufficient to prove that $x \;\mapsto\; x/{\approx}$ is a p-morphism from $\mathfrak{M}$ to $\mathfrak{M}/{\approx}$ reliable on Π. There are four things to check. (i) That this function is onto is clear. (ii) If $x\,R\,y$ then it follows from the definition of $R/{\approx}$ that $(x/{\approx})\,R/{\approx}\,(y/{\approx})$. (iii) Assume instead that $(x/{\approx})\,R/{\approx}\,(y/{\approx})$. Then there are x', y' such that $x' \approx x$, $y' \approx y$, and $x'\,R\,y'$. But since $y'/{\approx}$ is a cluster in $\mathfrak{M}^{x'}$ and $x \approx x'$, it easily follows from the definition of $\simeq$ that there must exist some cluster $z/{\approx}$ in $\mathfrak{M}^x$ such that $\mathfrak{M}^{y'} \simeq \mathfrak{M}^z$. (A rigorous proof of this assertion can be given by induction on the length of towers on x' in $\mathfrak{M}^{x'}$.) There must exist some $y'' \sim z$ such that $y' \equiv_\Pi y''$. That $x\,R\,y''$ is clear. That $y \approx y''$ follows from the fact that $y \approx y'$, $y' \equiv_\Pi y''$, and $\mathfrak{M}^y \simeq \mathfrak{M}''$. (iv) Finally, that $x \mapsto x/{\approx}$ is reliable on Π is obvious. □

Theorem 2.6.6. *For every positive natural number n, every normal extension of $\mathbf{K4B}_n$ has the finite model property.*

Proof. This follows at once from Theorems 2.6.2 and 2.6.5. □

This result generalizes Theorem 2.5.1, inasmuch as **K45** is a normal extension of $\mathbf{K4B}_2$. Among the consequences of Theorem 2.6.6 is the corollary that any normal extension of $\mathbf{S4.3B}_n$, for every n, has the f.m.p. This is our best approximation to Bull's Theorem, distinctly weaker than it. No model-theoretic proof of Bull's Theorem has yet been published, and it would be interesting to see one; perhaps the knowledge of such a proof would enable us to improve Theorems 2.6.1 and 2.6.6. However that be, Theorem 2.6.6 is a non-trivial result. Roughly speaking, it shows that non-finite logics with the property that every normal extension has the f.m.p. can be found arbitrarily near **K4**, provided **K4** is approached from the right direction.

2.6.3 Hereditary decidability of $\mathbf{K4.3B}_n$

Theorem 2.6.6 has an interesting consequence: every normal logic at least as strong as some $\mathbf{K4.3B}_n$ is decidable. We shall devote the rest of this section to the deduction of this result.

Lemma 2.6.7. *Suppose $\mathcal{C}$ is a class of finite indices of length n such that if $i_t = 0$ for some $i \in \mathcal{C}$ and $t < n$, then $j_t = 0$ for every $j \in \mathcal{C}$. Then there exists a finite class $\mathcal{D}$ of (not necessarily finite) indices of length n such that* $\mathsf{Logic}(\mathcal{C}) = \mathsf{Logic}(\mathcal{D})$.

Proof. By assumption, $n = \{0, \ldots, n-1\}$ is the index set for every index in $\mathcal{C}$. Let us say that $t < n$ is *unbounded* in $\mathcal{C}$ if, for every $m \in \mathbb{N}$, there is some $i \in \mathcal{C}$ such that $i_t > m$, and *bounded* in $\mathcal{C}$ otherwise. We shall agree to write $i(t_1/p_1, \ldots, t_k/p_k)$, where $p_1, \ldots, p_k$ are natural numbers or ω, for the index i' such that

$$i'_t \;=\; \begin{cases} i_t, & \text{if } t \notin \{t_1, \ldots, t_k\}, \\ p_1, & \text{if } t = t_1, \\ \cdots\cdots & \\ p_k, & \text{if } t = t_k. \end{cases}$$

We define $\mathcal{D}$ as the set of all indices i' of length n that satisfy these three conditions:

(1) There exists some $i \in \mathcal{C}$ such that $i' = i(t_{j_1}/\omega, \ldots, t_{j_q}/\omega)$, and $t_{j_1}, \ldots, t_{j_q}$ are unbounded in $\mathcal{C}$.

(2) For every $m \in \mathbb{N}$ there are $r_1, \ldots, r_q \in \mathbb{N}$ such that $r_1, \ldots, r_q > m$ and $i(t_{j_1}/r_1, \ldots, t_{j_q}/r_q) \in \mathcal{C}$.

(3) Let $u_1, \ldots, u_k$ be the unbounded elements of n other than $t_{j_1}, \ldots, t_{j_q}$. Then there exists some $m \in \mathbb{N}$ such that for all $r_1, \ldots, r_q \in \mathbb{N}$ and $s_1, \ldots, s_k \in \mathbb{N}$,

$$\text{if } r_1, \ldots, r_q > m \text{ and } i(t_{j_1}/r_1, \ldots, t_{j_q}/r_q, u_1/s_1, \ldots, u_k/s_k) \in \mathcal{C} \text{ then } s_1, \ldots, s_k < m.$$

To prove the lemma, it is enough to show:

(A) $\mathsf{Logic}(\mathcal{C}) = \mathsf{Logic}(\mathcal{D})$,
(B) $\mathcal{D}$ is finite.

Proof of (A). Suppose $i \in \mathcal{C}$. Then, by the definition of $\mathcal{D}$, there exists some $i' \in \mathcal{D}$ such that i and i' are alike except that some non-zero coordinates of i may be replaced by ω in i'. Hence $i \precsim i'$, so by Corollary 2.4.4, $\mathsf{Logic}(\mathcal{C}) \supseteq \mathsf{Logic}(\mathcal{D})$. To see that $\mathsf{Logic}(\mathcal{C}) \subseteq \mathsf{Logic}(\mathcal{D})$, suppose $A \notin \mathsf{Logic}(\mathcal{D})$, for some formula A. Then A is rejected by some model M on i, for some $i \in \mathcal{D}$. If $i \notin \mathcal{C}$, let Ψ be the set of subformulas of A. Let M' be the result of filtering only the infinite clusters of M through Ψ. This description is not precise, of course, but what is meant is the same procedure that was described in detail in the proofs of Theorems 2.5.1 and 2.6.1. Then A is rejected by M'. Moreover, the frame i' of M' is a finite index of length n, and $i'_t = 0$ only if $i_t = 0$. Therefore, by condition (2) of the definition of $\mathcal{D}$, there exists some $i'' \in \mathcal{C}$ such that $i' \precsim i''$. Hence $A \notin \mathsf{Logic}(\mathcal{C})$.

Proof of (B). There are at most n unbounded elements in n, so there are at most 2^n ways of selecting sets of unbounded elements in n. Let $t_1, \ldots, t_q$ be any fixed bounded elements in n. It will suffice to show that there can be only finitely many frames i' satisfying the three conditions of the definition of $\mathcal{D}$. Let us consider the possible coordinates i'_t, for $t < n$. If t is one of $t_1, \ldots, t_q$, then clearly there is no choice: $i'_t = \omega$. If t is bounded, then there is some bound $m(t)$, depending on t only, such that $i'_t < m(t)$, for then $i_t < m(t)$ for every index $i \in \mathcal{C}$. Finally, if t is unbounded and distinct from $t_1, \ldots, t_q$ then there is some bound $m(t_1, \ldots, t_q)$, depending only on $t_1, \ldots, t_q$, such that $i'_t < m(t_1, \ldots, t_q)$. For suppose that $i' \in \mathcal{D}$ and let i be the frame in $\mathcal{C}$ whose existence is granted by (1). Let $m(b_1, \ldots, b_p, t_1, \ldots, t_q)$ be the bound whose existence is granded by (3); here $b_1, \ldots, b_p$ are the coordinates of the bounded elements in n. By (2), we can find $r_1, \ldots, r_q > m(b_1, \ldots, b_p, t_1, \ldots, t_q)$ such that $i(t_1/r_1, \ldots, t_q/r_q) \in \mathcal{C}$. Since t is not among $t_1, \ldots, t_q$, $i_t = i(t_1/r_1, \ldots, t_q/r_q)_t$. By (3), then, $i_t < m(b_1, \ldots, b_p, t_1, \ldots, t_q)$. However, there are only finitely many possibilities for what integers $b_1, \ldots, b_p$ can be. Let $m(t_1, \ldots, t_q)$ be the largest possible bound of type $m(b'_1, \ldots, b'_p, t_1, \ldots, t_q)$. Then $i_t < m(t_1, \ldots, t_q)$, and this bound depends only on $t_1, \ldots, t_q$. $\square$

Theorem 2.6.8. *Every normal extension of* $\mathbf{K4.3B}_n$ *is the intersection of finitely many index logics.*

Proof. Let L be any normal extension of some $\mathbf{K4.3B}_n$. Then, by Theorem 2.6.6 and Corollary 1.3.8 and elementary reasoning, there is a class $\mathcal{C}$ of finite index frames of length $\leqslant n$ that determines L. Clearly $\mathcal{C}$ can be written as the union of finitely many pairwise disjoint sets of indices each of which satisfies the hypothesis of Lemma 2.6.7. The theorem therefore follows from that lemma. $\square$

Corollary 2.6.9. *Every normal extension of* $\mathbf{K4.3B}_n$ *is decidable.*

2.6.4 On axiomatization of index logics

If we showed that every index logic is axiomatizable, it would follow that every normal extension of $\mathbf{K4.3B}_n$ is axiomatizable (and even finitely axiomatizable). We shall not show here that every index logic is axiomatizable, but we shall prove the following general theorem about axiomatizations.

Theorem 2.6.10. *Let* $\Gamma = \Gamma(A_1, \ldots, A_m)$ *and* $\Delta = \Delta(B_1, \ldots, B_n)$ *be schemata in m and n schematic variables, respectively. Let* $L_1 = \mathbf{K4}\Gamma$ *and* $L_2 = \mathbf{K4}\Delta$. *Then* $L_1 \cap L_2 = \mathbf{K4}((\Gamma \wedge \Box\Gamma) \vee (\Delta \vee \Box\Delta))$.

Proof. To simplify matters, assume that $m = n = 1$; nothing in the proof will hinge on the number of schematic letters in Γ and Δ. Let us write

$$L = \mathbf{K4}(\,(\Gamma(A) \wedge \Box\Gamma(A)) \;\vee\; (\Delta(B) \wedge \Box\Delta(B))\,).$$

Since L_1 and L_2 are normal, $\Box\Gamma(A)$ is derivable in L_1 and $\Box\Delta(B)$ in L_2. It is clear, then, that $L \supseteq L_1 \cap L_2$.

To prove the converse inclusion, assume that A is any formula such that $A \notin L$. Then, by the Fundamental Theorem, there is some L-maximal set t of formulas such that $A \notin t$. Suppose that $A \in L_1$. Then t cannot be L_1-maximal. A simple inductive argument shows that if, for all formulas B, both $\Gamma(B) \in t$ and $\Box\Gamma(B) \in t$, then $L_1 \subseteq t$ (for L_1 contains **K4**) and hence t is L_1-maximal, contrary to what was just said. Therefore, there must exist some formula B such that $\Gamma(B) \wedge \Box\Gamma(B) \notin t$. But then, by L-maximality of t, it holds that, for every formula C, $\Delta(C) \wedge \Box\Delta(C) \in t$. By the same argument as before, $L_2 \subseteq t$. Hence t is L_2-maximal, so by the Fundamental Theorem $A \notin L_2$. Thus we have shown that $L \supseteq L_1 \cap L_2$. $\Box$

Corollary 2.6.11. *Let $\Gamma = \Gamma(A_1, \ldots, A_m)$ and $\Delta = \Delta(B_1, \ldots, B_n)$ be schemata. If $L_1 =$ **S4**Γ and $L_2 =$ **S4**Δ, then $L_1 \cap L_2 =$ **S4**$(\Box\Gamma \vee \Box\Delta)$.*

There is an obvious generalization of Theorem 2.6.10 to logics that are not extensions of **K4**. Roughly, the intersection of two logics **K**Γ and **K**Δ is axiomatized by adding to **K** every schema of type $\Box^m\Gamma \vee \Box^n\Delta$, for every $m, n \geqslant 0$. (This is not a finite axiomatization, of course.)

It would be interesting to know how much of our work on indices can be fruitfully generalized. What comes first to mind is to try and develop a theory for "partially ordered indices", which would be suitable for normal extensions of **K4** in the way our index theory is suitable for extensions of **K4.3**. One possible definition would be this: a triple $\langle X, \leqslant, f\rangle$ is a *partially ordered* (p.o.) *index* if (i) X is a set; (ii) $\leqslant$ is a partial ordering of X such that each chain in X is discrete, and for any $x, y \in X$ there is some $u \in X$ such that $u \leqslant x$ and $u \leqslant y$; and (iii) f is a function assigning to each element of X a non-negative integer or ω. Whether the introduction of such a concept would be of much help for the analysis of normal extensions of **K4** is not clear, and we shall not pursue the question further here.

2.7 Some particular systems

The exploration of modal logic has often been characterized by what one author has called a "stab-in-the-dark" strategy. A multitude of systems has been proposed, but often enough it seems to have been a matter of chance why one system was proposed and not another. It is always gratifying if a systematic analysis, although not designed to account for existing particular cases, in fact accounts for all or most cases in the literature and assigns to each its proper place, as it were. We shall consider, in this section, some extensions of **S4** brought forth by various authors and show that, whatever the original rationale for defining them, these logics are very easily described in terms of indices or p.o. indices.

2.7.1 The schema H

A schema due to Sobociński (see [Sobociński, 1964a]) is

H. $\quad A \to \Box(\Diamond A \to A)$.

It is usually discussed in the field of **S4**; the logic **S4H** is called **K1.2** by Sobociński. We shall extend our discussion slightly and analyze H in the field of **K4**.

Theorem 2.7.1. *If L is a normal extension of* **K4H**, *then its canonical model $\mathfrak{M}_L$ has this property:*

$$\text{if } \; x \, R_L \, y \; \text{ and } \; y \, R_L \, z \quad \text{then} \quad x = y \; \text{ or } \; y = z.$$

Proof. Assume there are x, y, z such that $x \, R_L \, y$ and $y \, R_L \, z$, but $x \neq y$ and $y \neq z$. Then there exist formulas A and B such that $A \in x$, $A \notin y$, $B \in z$, $B \notin y$. Hence

$$A \vee B \in x, \qquad A \vee B \notin y, \qquad A \vee B \in z.$$

Since $(A \vee B) \to \Box(\Diamond(A \vee B) \to (A \vee B))$ is an instance of H, we conclude that $\Box(\Diamond(A \vee B) \to (A \vee B)) \in x$. Since $\Diamond(A \vee B) \in y$, also $A \vee B \in y$. This is a contradiction. $\Box$

Theorem 2.7.2. *The logic* **K4H** *(***D4H***;* **S4H***) is determined by the class of all (serial; reflexive) transitive frames $\langle W, R\rangle$ such that if $x \, R \, y$ and $y \, R \, z$ then $x = y$ or $y = z$.*

It is easy to see that every frame for **K4H** has rank $\leqslant 2$. Hence, by Theorem 2.6.3, **K4H** is a normal extension of $\mathbf{K4B}_2$, so, by Theorems 2.6.2 and 2.6.5,

Theorem 2.7.3. *Every normal extension of* **K4H** *has the finite model property.*

This theorem would enable us to classify all the normal extensions of **K4H**. However, for simplicity we now restrict the scope to the normal extensions of **S4H**. Let us call a frame $F_n = \langle n, R\rangle$ a *fan* if $w\,R\,x$ iff $w = 0$ or $w = x$ (here $n \leqslant \omega$, and n is identified with the set $\{k \in \mathbb{N} \mid k < n\}$).

Theorem 2.7.4. *The logic* **S4H** *is determined by the countable fan F_ω. If L is a proper normal extension of* **S4H***, then L is determined by the fan F_n, for some positive integer n, and identical with* $\mathbf{S4HAlt}_n$.

Thus $\mathbf{S4HAlt}_1 \supset \mathbf{S4HAlt}_2 \supset \ldots \supset \mathbf{S4HAlt}_n \supset \ldots$ is a chain of strictly descending systems which contains every proper extension of **S4H** and whose intersection is **S4H**.

Proof. Let L be any normal extension of **S4H**. It follows from Theorem 2.7.3 that L is determined by a class $\mathcal{C}$ of fans. Observer that, for all $n \in \mathbb{N}$

(1) $\mathsf{Logic}(F_n) \supset \mathsf{Logic}(F_{n+1})$.

Indeed, the following defines a function from $n+1$ onto n:

$$f(k) \;=\; \begin{cases} k, & \text{if } k < n,\\ n-1, & \text{if } k = n.\end{cases}$$

If V is any valuation in F_n, then this defines a valuation in F_{n+1}:

$$V'(p_m) \;=\; \{\, k \in n+1 \mid f(k) \in V(p_m)\,\}, \qquad \text{for every } m \in \mathbb{N}.$$

It is easy to see that a formula rejected by the model defined by V on F_n, is also rejected by the model defined by V' on F_{n+1}. Thus, the inclusion (1) holds. That the inclusion is proper follows from the fact that Alt_n is derivable in $\mathsf{Logic}(F_n)$ but not in $\mathsf{Logic}(F_{n+1})$.

There is a similar proof for the assertion

(1) $\mathsf{Logic}(F_n) \supset \mathsf{Logic}(F_\omega)$.

Assume now that $\mathcal{C}$ is a finite class of fans. The there is a frame F_n with a larger number of elements than any other frame in $\mathcal{C}$. It follows from (1) that $\mathsf{Logic}(\mathcal{C}) = \mathsf{Logic}(F_n)$. Thus L is determined by the fan F_n, and it is clear that F_n determines the logic $\mathbf{S4HAlt}_n$ (see Lemma 1.5.3).

Assume instead that $\mathcal{C}$ is an infinite class of fans. By (2) we always have $\mathsf{Logic}(\mathcal{C}) \supseteq \mathsf{Logic}(F_\omega)$. From (1) and Theorem 2.7.3 we may infer that in the present case

$$\mathsf{Logic}(\mathcal{C}) = \mathbf{S4H}.$$

That F_ω is a frame for **S4H** is easily verified. Thus $\mathbf{S4H} \subseteq \mathsf{Logic}(\mathbf{F}_\omega)$. Consequently, in this case L is identical with **S4H** and hence determined by F_ω. □

The logic $\mathbf{S4HAlt}_1$ is of course an old acquaintance, the Trivial System. The logic $\mathbf{S4HAlt}_2$ is an interesting system: the index logic with index $\langle 1,1\rangle$. The letter system is known in the literature, although under a different axiomatization. Consider this axiom due to Sobociński (see [Sobociński, 1964a, Sobociński, 1970b]):

P. $\quad \Diamond\Box\Diamond A \to (A \to \Box A)$.

As Sobociński showed syntactically,

$$\mathbf{S4P} \;=\; \mathbf{S4HM}.$$

This result is easily corroborated by our semantic methods, and there are many other identifications derivable in the same way, for example,

$$\mathbf{S4P} \;=\; \mathbf{S4HAlt}_2 \;=\; \mathbf{S4.3H} \;=\; \mathbf{S4.2MAlt}_2 \;=\; \mathbf{S4MB}_2.$$

The Trivial System and **S4P** are the only normal extensions of **S4.3**.

2.7.2 The schema Zem

We next consider the following schema, which was first defined in [Zeman, 1968]:

P. $\Box\Diamond\Box A \to (A \to \Box A)$.

S4Zem is known in the literature as **S4.04**.

Theorem 2.7.5. *The logic* **S4Zem** *is determined by the class of all finite reflexive frames of rank* $\leqslant 2$ *such that the initial cluster is simple.*

Proof. The consistency part is no problem. For the completeness part we shall have recourse to filtration theory. Let $\mathfrak{M} = \langle W, R, V\rangle$ be any point generated submodel of the canonical model for **S4Zem**. Let Ψ be a set of formulas closed under subformulas and logically finite in **S4Zem**. To prove the theorem, it will suffice to show that the Lemmon filtration $\mathfrak{M}' = \langle W', R', V'\rangle$ of $\mathfrak{M}$ through Ψ, which is of course finite, is of rank $\leqslant 2$ and has a simple initial cluster. Suppose t generates $\mathfrak{M}$. Then $[t]$ belongs to the initial cluster of $\mathfrak{M}'$. Let $C_1, \ldots, C_n$ be all final non-initial clusters in $\mathfrak{M}'$ (the non-initial proviso applies only when $\mathfrak{M}'$ happens to be of rank 1; if so, $n = 0$). For each i such that $1 \leqslant i \leqslant n$ let $C_i(1), \ldots, C_i(k_i)$ be all clusters immediately preceding C_i. For each j such that $1 \leqslant j \leqslant k_i$ there must exist $B_{i,j} \in \Psi$ such that $\Box B_{i,j}$ is true in $\mathfrak{M}'$ at every point in C_i and at no point in or preceding $C_i(j)$. It follows by Theorem 1.7.7 that $\Box B_{i,1} \vee \ldots \vee \Box B_{i,k_i}$ is true at w in $\mathfrak{M}$ if and only if $[w] \in C_i$. Let $B_i = B_{i,1} \wedge \ldots \wedge B_{i,k_i}$. Evidently,

$$\mathfrak{M}, w \models \Box B_i \quad \text{if and only if} \quad [w] \in C_i.$$

Consequently,

$$\mathfrak{M}, w \models \Box B_1 \vee \ldots \vee \Box B_n \quad \text{if and only if} \quad \text{the cluster } [w]/\!\sim \text{ is final in } \mathfrak{M}'.$$

We know (for example from the proof of Lemma 2.2.1) that there can be found a Boolean combination A of formulas in Ψ such that A is true in $\mathfrak{M}'$ at $[t]$, and at $[t]$ only. Thus, by Theorem 1.7.7,

$$\mathfrak{M}, w \models A \quad \text{if and only if} \quad [w] = [t].$$

It follows that

$$\mathfrak{M}, t \models (A \vee \Box B_1 \vee \ldots \vee \Box B_n);$$
$$\mathfrak{M}, t \models \Box\Diamond\Box(A \vee \Box B_1 \vee \ldots \vee \Box B_n).$$

Hence, by using the force of the new schema Zem, we infer

$$\mathfrak{M}, t \models \Box(A \vee \Box B_1 \vee \ldots \vee \Box B_n).$$

Let $[w]$ be any element of $\mathfrak{M}'$. If $[w] \neq [t]$, then $\mathfrak{M}, w \not\models A$. But then also $w \neq t$, so

$$\mathfrak{M}, w \models (A \vee \Box B_1 \vee \ldots \vee \Box B_n).$$

Hence $[w]/\!\sim$ is final. $\Box$

It would not be difficult to prove – although we shall not give the proof here – that there it a single frame that also determines **S4Zem**: the frame of rank 2 which has a simple initial cluster and denumerably many denumerable final clusters. More formally this frame could be described as the frame $\langle W, R\rangle$, where

$$W = \{ n \in \mathbb{N} \mid n = 1 \text{ or } n = p^r, \text{ for some prime } p \text{ and some } r \geqslant 1 \};$$
$$m\,R\,n \quad \text{if and only if} \quad m \mid n \ \text{ or } \ n \mid m.$$

It follows from this that the schema B_2 is valid in **S4Zem** and, thus, that **S4Zem** is an extension of $\mathbf{K4B}_2$. By Theorems 2.6.2 and 2.6.5, then:

Theorem 2.7.6. *Every normal extension of* **S4Zem** *has the finite model property.*

In principle this theorem can be used to classify completely the normal extensions of **S4Zem**. The situation is not so simple as in the case of **S4H**, though: the extensions do not form a chain, and it is not even true that every extension is monolithic. For example, let F_1 and F_2 be the subframes of the frame for **S4Zem** mentioned above where the domain of F_1 is $\{1, 2, 4, 3, 9\}$ and the domain of F_2 is $\{1, 2, 3, 9, 27\}$. Then $\mathsf{Logic}(F_1)$, $\mathsf{Logic}(F_2)$, and $\mathsf{Logic}(\{F_1, F_2\})$ are normal extensions of **S4Zem**, but $\mathsf{Logic}(F_1)$ and $\mathsf{Logic}(F_2)$ are incompatible, and there is no point generated frame that determines $\mathsf{Logic}(\{F_1, F_2\})$.

If the class of normal extensions of **S4Zem** is a bit too complicated to plot in a chart, the class of normal extensions of **S4.3Zem** is quite simple. We exhibit its structure in Figure 2.2, and we sketch the considerations that lead to it. It follows from Theorem 2.7.6 and elementary reasoning that every normal extension of **S4.3Zem** is determined by a class $\mathcal{C}$ of finite index frames of rank $\leqslant 2$. The following list exhausts all possibilities.

(1) $\mathcal{C}$ is finite.

- (α) Every frame in $\mathcal{C}$ is of rank 1. Then, by a corollary to the Index Theorem, there is some $n \in \mathbb{N}$ such that $\mathsf{Logic}(\mathcal{C}) = \mathsf{Logic}(n)$.
- (β) Every frame in $\mathcal{C}$ is of rank 2. Then, similarly, there exists some $k \in \mathbb{N}$ such that $\mathsf{Logic}(\mathcal{C}) = \mathsf{Logic}(\langle 1, k\rangle)$.
- (γ) There are frames of both ranks in $\mathcal{C}$. Let n be the largest integer such that the index frame $n \in \mathcal{C}$, and let k be the largest integer such that the index frame $\langle 1, k\rangle \in \mathcal{C}$. If $n \leqslant k$ then $\mathsf{Logic}(\mathcal{C}) = \mathsf{Logic}(1, k)$. If $n > k$ then $\mathsf{Logic}(\mathcal{C}) = \mathsf{Logic}(n) \cap \mathsf{Logic}(\langle 1, k\rangle)$.

(2) $\mathcal{C}$ is infinite.

- (δ) There are finitely many frames of rank 2 in $\mathcal{C}$. Let k be the largest integer such that the index frame $\langle 1, k\rangle \in \mathcal{C}$. Then $\mathsf{Logic}(\mathcal{C}) = \mathbf{S5} \cap \mathsf{Logic}(\langle 1, k\rangle)$, since, as we know, $\mathsf{Logic}(\omega) = \mathbf{S5}$.
- (ϵ) There are infinitely many frames of rank 2 in $\mathcal{C}$. Then $\mathsf{Logic}(\mathcal{C}) = \mathbf{S4Zem}$, since, as is easily seen, $\mathbf{S4Zem} = \mathsf{Logic}(\langle 1, \omega\rangle)$.

In the chart, Figure 2.2, systems are marked, as usual, by open spheres. The index logics have their index beside them: all the other logics are intersections of index logics and not monolithic. Notice that there are no intercalary logics where the lines are unbroken, whereas there are denumerably many logics where the lines are broken.

Figure

Figure 2.2: Figure.

Triv, **S5**, **S4P**, and **S4.3Zem**, although the only logics named in the diagram, are not the only systems that have names. Thus, for every $n \geqslant 2$,

$$\mathbf{S4Alt}_n \;=\; \mathbf{S4P} \,\cap\, \mathbf{S5Alt}_n.$$

In particular, $\mathbf{S4Alt}_2 \;=\; \mathbf{S4P} \cap \mathbf{S5Alt}_2$. The latter system is identical with Sobociński's system **V1** (see [Sobociński, 1964a, Sobociński, 1964b]), which is got by adding to **S4** the schema

V. $\quad \Box A \vee \Box(A \to B) \vee \Box(A \to \neg B)$.

(That Alt_2 and V are equivalent in **K**, and a fortiori in **S4**, is easy to prove even syntactically.)

However, there is a more interesting celebrity hiding incognito in Figure 2.2: $\mathbf{S4P} \cap \mathbf{S5}$. It seems that its true identity has never been revealed before, and it is a pleasure to do so here. Consider the schema, related to 5 and V:

Sch. $\quad \Box(\Diamond\Box A \to \Box A) \vee \Box B \vee \Box(B \to C) \vee \Box(B \to \neg C)$.

This schema is proposed in [Sobociński, 1970b], but we name it in honor of Schumm who was the first to consider the logic **S4Sch** (see [Schumm, 1969a]). Schumm defined **S4Sch** in response to a question of Sobociński whether there are systems stronger than the logic we write **S4.3Zem** and weaker than **S5**. It is interesting that Schumn was able to find the strongest system with this property, for

$$\mathbf{S4Sch} \;=\; \mathbf{S4P} \cap \mathbf{S5},$$

as is readily seen from Corollary 2.6.11. Since any of the following schemata is equivalent in **S4** to Sch, it might as well have been added to **S4** in order to get **S4Sch**:

$$\begin{array}{l}(\Diamond\Box A \to \Box A) \vee \Box B \vee \Box(B \to C) \vee \Box(B \to \neg C)\\ (\Diamond\Box A \to A) \vee \Box B \vee \Box(B \to C) \vee \Box(B \to \neg C)\\ (\Diamond\Box A \to A) \vee (\Diamond\Box\Diamond B \to (B \to \Box B))\end{array}$$

There are of course indefinitely many other schemata of this sort.

2.7.3 The schema R

We now turn to the study of the schema

R. $\quad \Diamond\Box A \to (A \to \Box A)$.

It seems that this schema appeared in print for the first time in [Sobociński, 1964b]. Later it has been discussed in [Bull, 1967], where it is attributed to Geach. Bull gave a completeness theorem for **S4R**, couched in algebraic terms. One in model-theoretic terms is this:

Lemma 2.7.7. *The logic* **S4R** *is determined by the class of frames* $\langle W, R\rangle$ *satisfying this condition: if* $x \neq z$ *and* $x\,R\,z$*, then* $x\,R\,y$ *implies that* $y\,R\,z$*.*

Proof. Consistency is no problem. Let $\mathfrak{M} = \langle W, R, V\rangle$ be the canonical model for **S4R**. Take any distinct x and z such that $x\,R\,z$. Then there is some A such that $A \in x$ and $A \notin z$. Let y be any element such that $x\,R\,y$. Take any formula B such that $\Box B \in y$. Then also $\Box(A \vee B) \in y$, so $\Diamond\Box(A \vee B) \in x$. Since clearly $(A \vee B) \in x$, it follows from the new schema R that $\Box(A \vee B) \in x$. Hence $B \in z$. This proves that $y\,R\,z$. $\Box$

Theorem 2.7.8. *The logic* **S4R** *is of index* $\langle 1, \omega\rangle$*.*

Proof. Clearly, $\langle 1, \omega\rangle$ is a frame for **S4R**. The result then follows from Lemma 2.7.7 and the Index Theorem. $\Box$

Corollary 2.7.9. **S4R** = **S4.3Zem**.

S4R is known in the literature as **S4.4**. This means that we have answered another open question: there is no intermediate system between **S4.4** and **S5** that is not at the same time included in **S4P**. ([Sobociński, 1970b, p. 367]. **S4P** is called **K4** by Sobociński.)

2.7.4 The schema F

Yet another schema, introduced in [Zeman, 1968], is

F. $\quad \Box(\Box A \to B) \vee (\Diamond\Box B \to A)$.

Lemma 2.7.10. *The logic* **S4F** *is determined by the class of frames* $\langle W, R\rangle$ *satisfying this condition: if* $x\,R\,z$ *and not* $z\,R\,x$*, then* $x\,R\,y$ *implies that* $y\,R\,z$*.*

Proof. As usual, consistency offers no problem. Let $\mathfrak{M} = \langle W, R, V\rangle$ be the canonical model for **S4F**. Suppose x and z are elements in $\mathfrak{M}$ such that $x\,R\,z$ but not $z\,R\,x$. Then there is some formula A such that $A \notin x$ and $\Box A \in z$. Take any y such that $x\,R\,y$. Let B be any formula such that $\Box B \in y$. Then $\Diamond\Box B \in x$, so $(\Diamond\Box B \to A) \notin x$. Therefore, by the new schema F, $\Box(\Box A \to B) \in x$. Consequently $B \in z$. This proves that $y\,R\,z$, which is what we want. $\Box$

Theorem 2.7.11. *The logic* **S4F** *is of index* $\langle \omega, \omega\rangle$*.*

Proof. Clearly, $\langle \omega, \omega\rangle$ is a frame for **S4F**. The result then follows from Lemma 2.7.10 and the Index Theorem. $\Box$

Figure

Figure 2.3: Figure.

It follows that $\mathbf{S4F} = \mathbf{S4.3B}_2$.

Let us review the extensions of **S4F** that have been discussed in this section. In Figure 2.3 we extend the picture given by Figure 2.2. Every sphere represents a system; the filled spheres stand for systems that are found in the literature. As far as the author knows there are no others than those we have indicated. Apart from **S5** and its extensions they are, with our notation to the left and that of the Sobociński school to the right:

$$\begin{array}{rcl} \mathbf{S4P} & = & \mathbf{K4} \\ \mathbf{S4P} \cap \mathbf{S5Alt}_2 & = & \mathbf{V1} \\ \mathbf{S4P} \cap \mathbf{S5} & = & \mathbf{S4.7} \;\; (= \;\; \mathbf{S4.6} \;\; = \;\; \mathbf{S4.5}) \\ \mathbf{S4R} & = & \mathbf{S4.4} \\ \mathbf{S4FM} & = & \mathbf{K3.2} \\ \mathbf{S4F} & = & \mathbf{S4.3.2} \end{array}$$

It is not clear what reasons were instrumental when these systems were first defined. In all likelihood, though, they were not semantical. We find it quite interesting, therefore, that among the infinity of systems those have been singled out that “stand out” in one way or other.

Figure

Figure 2.4: Figure.

Whereas Figure 2.3 is complete as far as normal extensions of **S4R** go (accepting that infinitely many systems are intimated rather than individually represented), no effort has been made to sort out the interrelations of all the normal extensions of **S4F**. That it presents a problem to draw a complete picture which does not fade into the incomprehensible is shown already by the fair complexity of Figure 2.4, in which are represented all normal extensions of the logic with index $\langle 2, 2\rangle$. For each logic, we have indicated the class of index frames that determines it. For comparison, we have plotted, in Figure 2.5, all normal extensions of the index logic $\langle 3, 3\rangle$. Here the complexity has grown so great that in order to ensure legibility we have only marked the index logics. However, the reader will have no difficulty in finding the classes of indices determining the non-monolithic logics.

Figure

Figure 2.5: Figure.

Let us remark, finally, that Bull’s Theorem yields a Scroggs’ Theorem for **S4.3Grz**. For it is readily seen that if L is a normal extension of **S4.3Grz**, then every frame for L is a discrete linear ordering. By Bull’s Theorem then, every normal extension of **S4.3Grz** is determined by a class of index frames of index of type 1^n. Using the Index Theorem, we conclude that any normal extension is either identical with $\mathbf{S4.3GrzAlt}_n$ and determined by 1^n, for some natural number $n \geqslant 1$, or else it is the improper extension. **S4.3Grz** itself has index ω^*, so everything works out neatly. (Note that this dissertation in not self-contained on this point as we have not given a proof of Bull’s Theorem here.)

2.8 Historical remarks

The logic **S4** and its extensions have been favorite objects of study ever since the time of C. I. Lewis, and they belong to one of the best-known areas of modal logic. The interest in extensions of **K4** that are not also extensions of **S4** has been much smaller. In this section we shall make a few remarks on the history of the material presented in this chapter, in order to supplement the references given in the text.

The main results of Section 2.2 are reported in [Segerberg, 1970a]; the special results for strict linear orders, of interest to tense logic, are proved in [Segerberg, 1971]. Here it should be mentioned that the first published completeness proof for the logic **D4.3Z** is due to P. Schindler; see [Schindler, 1970]. A readable account of early work on tense logic, and, in particular, of the search for the so-called Diodorean system – our **S4.3Dum** – is given in [Prior, 1967]. In [Bull, 1965] Kripke is credited with having given the first completeness proof for **S4.3Dum**; Bull himself gives an algebraic proof and establishes the f.m.p. The schema Grz, which the author has elsewhere erroneously attributed to Grzegorczyk, appears to have been discussed for the first time in [Sobociński, 1964a]. The rationale for sticking to the name "Grz" is that **S4Grz** is identical with the interesting system in [Grzegorczyk, 1967]; one reason is also that Sobociński has introduced so many new schemata that it does not seem right to attach his name to only one of them. The index logic 1^{ω^*} – as we would call it – was discussed in [Makinson, 1966b], but it was only in [Schumm, 1969b] that it was identified as **S4.3Grz**. The same result, along with the other main results of Section 2.3, was announced in [Segerberg, 1970b]. That abstract however, contains an error regarding **S4Dum**, which is corrected in [Segerberg, 1970a].

There is much research going on in this area, and there is the following to say about unpublished results, as far as they are known by the author. G. Schumm has informed me, in letters dated October 2 and November 4, 1970, that he has independently analysed the systems we call **S4Grz**, **S4R**, **S4P**, **S4F**, and **S4FM**. Schumm also mentions that he has "mapped out all the normal extensions of **S4.4**", which presumably means that he has a chart equivalent to our Figure 2.3. Kit Fine has informed obtained a model-theoretic proof of Bull's Theorem (for which this author has long been looking, in vain). He also claims to have shown that all (normal?) extensions of **S4.3** are finitely axiomatizable and, hence, decidable.

Results not attributed to others, in this section or in the main text, should be new.

Chapter 3

Quasi-normal systems

Chapter 4

Regular and quasi-regular systems

4.1 Examples of regular logics

4.2 Relations between normal and regular logics I

4.3 Relations between normal and regular logics I

4.4 Quasi-regular systems I

4.5 Quasi-regular systems II

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Changes to text

The xero-copy of the original text we had was not of high quality, many symbols were faded and hardly readable. Even the ABBYY FineReader had difficulties in recognizing the text on some pages. Our aim of typesetting was not only to obtain a high-quality text, but mainly to "decipher" the unreadable pieces of the text (in order to do so, the reader has to have read several pages prior to the page under investigation, to be able to guess mathematical symbols). Additionally, we tried to adopt the contemporary notation and slightly changed the formatting by using the advantage of LaTeX. Finally, some typos were corrected (see LaTeX sources for marks).

- <u>Underlined</u> text is turned into *italized.*
- Sections in our text are numbered 1.1, 1.2, 1.3, etc.; not just 1, 2, 3, etc.
- We added subsections (numbered as 1.1.1, 1.1.2, etc.) to more structure the text.
- We removed the words "This ends the proof." (the original text did not have a symbol for the end of the proof, but we have).
- Names of modal systems are boldfaced: **K4**, **S5**; in the original text, they are not distinguished.
- Names of schemata (modal formulas) are sans-serif: K, Q, S, etc.
- The set of natural numbers is denoted by $\mathbb{N}$, not Nat.
- The set of integer numbers is denoted by $\mathbb{Z}$, not Int.
- The set of all modal formulas is denoted by Fm, not Φ.
- The "by definition" sign is $:=$, not $=_{\mathrm{df}}$.
- We write propositional letters as p_n, not P_n.
- The set of possible worlds is denoted by W, not U; a typical world is $w \in W$, not $u \in U$.
- A (neighborhood or relational) model is denoted by M, not $\mathfrak{U}$.
- A frame is denoted by F, not $\mathcal{F}$.
- A valuation is a map $V\colon \mathsf{Var} \to \wp(W)$, not $V\colon \mathbb{N} \to \wp(W)$. So, $V_{\mathrm{new}}(p_n) := V_{\mathrm{old}}(n)$.
- We write $M, w \models A$ instead of $\models_w^M A$.
- The canonical model of a logic L is denoted by $\mathfrak{M}_L$.
- On pp.61–62 (Hintikka schema and condition) we corrected several mistakes and improved notation.
- Segerberg denoted by R^* the *ancestral* of R, which, in modern terminology, corresponds to the transitive closure of R. We denote R^+ the transitive closure and R^* the reflexive-transitive closure of R.
- What Segerberg called "generated submodel", we now call "point generated submodel" (if one reads some piece in the middle of the dissertation and encounters "generated submodel", the reader will hardly guess that the author means "generated by some point", because nowadays, the notion "generated submodel" is also used, in a different meaning (generated by some subset of points)).
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