Aperiodic tilings by right triangles *

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Abstract

Let ψ denote the square root of the golden ratio, $\psi = \sqrt{(\sqrt{5}-1)/2}$. A golden triangle is any right triangle with legs of lengths a, b where $a/b = \psi$. We consider tilings of the plane by two golden triangles: that with legs $1, \psi$ and that with legs ψ, ψ^2. Under some natural constrains all such tilings are aperiodic.

1 Introduction

1.1 Golden triangles

The altitude of every right triangle cuts it into two similar triangles. Are there other polygons P that can be divided into two polygons each of which is similar to P? Any parallelogram whose width is $\sqrt{2}$ times bigger than its length has this property: its median cuts it into two equal such parallelograms. A more interesting example is the so called *Ammann hexagon*[1]:

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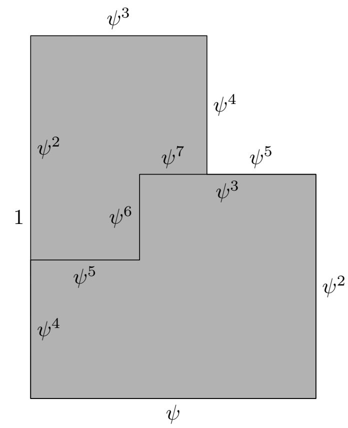
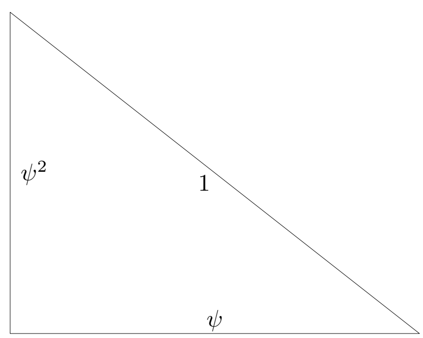
[1]This hexagon is attributed to Robert Ammann in [8]. Independently the hexagon was discovered by Scherer [6] who called it the *Golden Bee*.

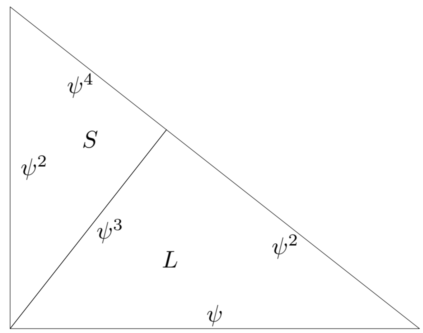


Here ψ stands for the square root of the golden ratio ($\psi^4 + \psi^2 = 1$). It turns out that that there are no other such hexagons. This was conjectured by Scherer in [6] and proved by Schmerl in [7].

Let us go back to the right triangle, whose altitude cuts it into two similar triangles. Assume that its hypotenuse and legs are proportional to 1, ψ and ψ^2, respectively, so that Pythagorean theorem hold:



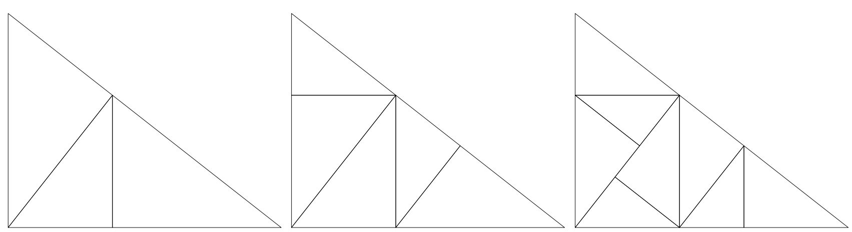
Call any such triangle a *golden triangle*. The altitude of the golden triangle cuts it into a *large* and *small* golden triangles, labeled by letters “L” and “S”:



The ratio between the sizes of the initial triangle and its large part is equal to the ratio between the sizes of its large and small parts (hence the name).

1.2 Standard tilings

Let us start with a golden triangle and cut it by its altitude into two smaller golden triangles. Then cut the larger of the resulting triangles into two triangles. We obtain one small triangle and two large triangles. Then again cut both large resulting triangles, then again and again .... We get the following tilings:



...

On each step we obtain a tiling of the original golden triangle by golden triangles of two sizes. We will call such tilings *standard*. The number of steps needed to obtain a standard tiling from the original golden triangle is called its *depth*. For example, the depth of the last tiling in the last picture is 4.

We can start from any golden triangle, so for each n and each d we can obtain a standard tiling of depth n consisting of triangles of sizes d and ψd. In this paper, we study tilings of the plane (or its parts) by golden triangles of two sizes d and ψd (where d is a fixed number, say, 1) that look locally like standard tilings. This means that for any circular window (of any diameter D) every pattern that we can observe in that tiling through such a window can be observed also in some standard tiling. Of course the depth of that tiling may depend on the diameter of the window. The larger the window is the larger the depth of the tiling may be. We will call such tilings *locally standard, LS.* In other words, a tiling is LS if each its finite subset is a subset of a standard tiling. (Throughout the paper we consider tiling as sets of triangles.)

Do locally standard tiling of the plane exist? This can be shown by well known arguments (used in the literature, for instance, for Berger's tilings [2]). Moreover, like Berger's tilings, all locally standard tilings of the plane are aperiodic.

Assume now that we bound the size of the window by some constant D. That is, consider only patterns of diameter at most D. Our Theorem 2 states that for any D there are finitely many patterns of diameter at most D that can be observed in standard tilings. (When counting patterns, we identify isometric ones.)

More specifically, we say that a finite tiling T is a *pattern* of a tiling T' if T is a subset of T'. For example, every standard tiling is a pattern of every standard tiling of larger depth, but not the other way around. A pattern is *standard* if it is a pattern of a standard tiling. Thus a tiling is LS iff all its patterns are standard. The *diameter of a pattern* is the maximal distance between two points lying in triangles of that pattern. Theorem 2 states that for every D there are finitely many standard patterns of diameter at most D.

Theorem 1 gives a hope to describe locally standard tilings by a finite number of patterns. This would be possible if there were D with the following property: if all patterns of diameter at most D of a tiling T are standard then T is locally standard (that is, all patterns of T are standard).

The main result of the paper, Theorem 3, states that this is not the case. In other words, for every D there is a tiling of the plane by golden triangles that is not LS and yet all its patterns of diameter at most D are standard. Speaking informally, locally standard tilings cannot be finitely presented, they cannot be defined by a finite set of local rules.

This result shows a crucial difference between tilings by Ammann hexagons and golden triangles. Recall that Ammann hexagon of size d can be also cut into two Ammann hexagons of sizes ψd and $\psi^2 d$ (see the picture on the first page). In the similar way one can define *Ammann standard tilings*, *Ammann locally standard tilings* etc. However, this time there is D such an Ammann tiling of the plane is locally standard iff all its patterns of diameter at most D are standard [3]. Moreover, an Ammann tiling is locally standard iff all its pairs of adjacent hexagons form a standard pattern!

Let us return to tilings by golden triangles. There is yet another way to define what means that a tiling of a plane "looks like standard tilings". Let us call the operation used to define standard tiling *the refinement*. The refinement of a tiling T is the tiling obtained from T by cutting each large triangle from T by its altitude (and keeping all small triangles intact). It is not hard to see that different tilings have different refinements. Hence a reverse partial operation is well defined. That partial operation is called the *coarsening*. (Not every tiling has a coarsening: for example, a tiling consisting of one small triangle has no coarsening.) If a tiling admits n successive coarsenings, we call it *n-coarsenable*. For instance, any standard tiling of depth n is $n+1$-coarsenable but not $n+2$-coarsenable. If a tiling is n-coarsenable for all n we call it *infinitely coarsenable, IC*. One can show that every LS tiling is IC, but not the other way around. An example of IC tiling which is not LS will be given later. From this example it will be clear that the class of LS tilings is a more adequate formalization of tilings that "look like standard tilings" than the class of IC tilings.

Our main result applies to the class of IC tilings as well: IC tilings cannot be defined by a finite set of local rules. More specifically, for any D there is a non-IC tiling whose all patterns of diameter at most D are standard (and hence appear in an IC tiling, namely, in any LS tiling of the plane).

One can wonder if the class of locally standard tilings is "sofic". We say that a class $\mathcal{C}$ of tilings is *sofic* if the following holds. There are D, a finite set of colors and a finite set of patterns $\mathcal{P}$ of diameter at most D where each pattern consists of colored triangles (each triangle bears only one color) such that

(1) in every tiling from $\mathcal{C}$ each triangle can be colored so that all patterns of diameter at most D of the resulting colored tiling belong to $\mathcal{P}$, and the other way around:

(2) if every pattern of diameter at most D in a tiling of the plane by colored triangles belongs to $\mathcal{P}$, then after removing colors the resulting tiling belongs $\mathcal{C}$.

We do not know whether the families of LS tilings and of IC tilings are sofic. The Goodman-Strauss theorem [5], or its proof, might provide a positive answer to this question. The statement of that theorem itself does not imply the answer, as its conditions are not satisfied for the family of LS (or IC) tilings. The same applies to Fernique – Ollinger generalization of Goodman-Strauss theorem [4].

2 Preliminaries

The letter ψ denotes the square root of the golden ratio, $\psi = \sqrt{(\sqrt{5}-1)/2}$. A *golden triangle* is any right triangle similar to that shown on the picture on page 2 (all points inside the triangle are considered as belonging to it). The *size* of a golden triangle is the length of its hypotenuse. A *d-tiling* is a non-empty set of golden triangles that pair wise have no common interior points and each of them is either of size d (such triangles are called *large*), or of size $d\psi$ (those are called *small*).

A *tiling* is a d-tiling for some d. A tiling T *tiles* A (where A is a subset of the plane), if A equals the union of all triangles in T. A tiling T is called *periodic*, if there is a nonzero vector v (called a *period*) such the result of transition of every triangle H in T by vector v belongs to T. Otherwise the tiling is called *aperiodic*.

The *refinement* of a d-tiling T is the ψd-tiling obtained from T by cutting each large triangle from T by its altitude. All small triangles remain intact and become large triangles of the refinement. It is easy to verify that the refinement is an injective operation. The reverse partial operation is called the *coarsening*. The *k-refinement* of a tiling is the result of applying k successive refinements to it. The partial operation of *k-coarsening* is defined in a similar way. If a tiling admits k successive coarsenings, that is, it is a k-refinement of some tiling, we call it *k-coarsenable*. If a tiling is k-coarsenable for all k we call it *infinitely coarsenable*.

A *standard d-tiling of depth n* is a d-tiling obtained from a single golden triangle H of size $d\psi^{-n}$ by n successive refinements.

A finite tiling P is a *pattern* of a tiling T if P is a subset of T. A finite tiling is a *standard pattern* if it is a subset of a standard tiling. A tiling T is *locally standard* all its patterns are standard, i.e. for all finite $W \subset T$ there is a standard tiling T' with $W \subset T'$.

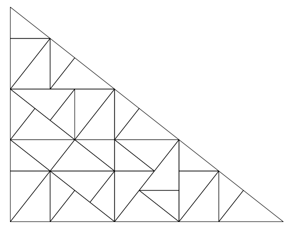
The *diameter* of a finite tiling is the maximal distance between two points lying in triangles of that tiling. A tiling is called *D-locally standard* if all its patterns of diameter at most D are standard.

3 Results

Theorem 1. *(a) There are locally standard tilings of the plane. (b) Every locally standard tiling of the plane is infinitely coarsenable. (c) The converse is not true. (d) Every infinitely coarsenable tiling of the plane is aperiodic.*

Proof. (a) Let St_n denote the standard 1-tiling of depth n. Observe that

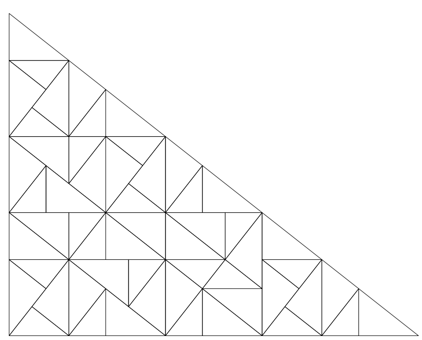
St_7 has a large triangle that is located strictly inside the part of the plane tiled by St_7:



This implies that we can draw St_0 and St_7 on the plane so that St_0 is a subset of St_7 and, moreover, the triangle forming St_0 is strictly inside the part of the plane tiled by St_7. Similarly, we can draw St_{14} on the plane so that St_{14} includes St_7 as a subset and the part of the plane tiled by St_7 is strictly inside the part of the plane tiled by St_{14}.

In this way we can construct a sequence of tilings $\mathrm{St}_0, \mathrm{St}_7, \mathrm{St}_{14}, \dots$ such that St_{7n} is a standard tiling of depth $7n$, St_{7n} is a subset of St_{7n+7} and the union $T = \bigcup_{n=0}^{\infty} \mathrm{St}_{7n}$ tiles the entire plane. On the other hand, the tiling T is locally standard by construction.

(c) In the same way as in item (a), we can construct a sequence of tilings $\mathrm{St}_0, \mathrm{St}_8, \mathrm{St}_{16}, \dots$ such that the union $T = \bigcup_{n=0}^{\infty} \mathrm{St}_{8n}$ tiles a half-plane. This is because the tiling St_8



has a large triangle L such that the hypotenuse of L lies on the hypotenuse of the triangle tiled by St_8 and both legs of L are strictly inside the triangle tiled by St_8.

Let $\tilde{T}$ be the tiling obtained from T by applying the axial symmetry with the axis equal to the edge of the half plane tiled by T. Both T and $\tilde{T}$ are IS by construction. Then shift $\tilde{T}$ by a very small amount along the edge of the half plane. The shifted $\tilde{T}$ is IC as well. Hence the union of T and shifted $\tilde{T}$ is also IC. On the other hand, it is not LS, as all patterns along the edge of the half plane become non-standard after the shift.

(b) We will say that a small triangle S and a large triangle L form a *couple* if they are located as shown on the second picture on page 3. It is easy to see that in any standard tiling for any small triangle S there is a large triangle L forming a couple with S.

Consider any small triangle S in a locally standard d-tiling T of the plane. As T is locally standard, there is a large triangle $L \in T$ forming a couple with S. Replace in T the triangles L and S by their union $L \cup S$, for every small triangle S. We obtain a $\psi^{-1}d$-tiling T', whose refinement equals T. Thus T is coarsenable.

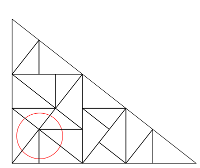
Let us show that T' is LS. Let W' be any finite subset of T'. We have to show that W' is a subset of a standard tiling. Let W stand for the refinement of W'. Then W is a subset of T. Since T is LS, W is a subset of a standard tiling, say, St_n. If $n = 0$ then W' is a small triangle and we are done. Otherwise W' is a subset St_{n-1}.

(d) Assume that an IC tiling T has a non-zero period v. Then v is also a period of the coarsening T' of T. Indeed, the refinement of $T' + v$ is equal to the tiling $T + v$, which equals T by the assumption; thus $T' + v$ and T' have the same refinement and hence coincide. Similarly, v is a period of the coarsening T'' of T' and so on. Note that the coarsening increases the sizes of triangles. Thus, on some step, v becomes smaller then the lengths of all sides of triangles and we get a contradiction. □

Theorem 2. *For any D, d the family of patterns of diameter at most D of standard d-tilings is finite. (When counting patterns we identify isometric ones.)*

Proof. Call a tiling P a *simple* pattern of a tiling T if there is a node K of some triangle from T (called the *center* of the pattern) such that P consists of all the triangles from T whom K belongs to.

A *simple standard pattern* is a simple pattern of a standard tiling. An example of a simple standard pattern P is shown on the following picture (P consists of all triangles intersecting the circle, the center of the pattern is inside the circle):



The proof is based on the following two lemmas.

Lemma 1. *Assume that a* 1*-tiling* T *tiles a convex set* U*. Assume further that* S *is a subset of* U *of diameter less than a certain positive constant* ε*. Then* T *has a simple pattern* P *that covers* S*.*

Lemma 2. *For every* d *the family of all simple patterns of* d*-tilings is finite and their number does not depend on* d*. (We identify here isometric patterns.)*

Both lemmas are quite technical and will be proved in the Appendix. Now we finish the proof of the theorem assuming the lemmas. Fix D and d. W.l.o.g. assume that $d = 1$. Consider a standard 1-tiling St_n of some depth n. Let W be any pattern of diameter at most D of the tiling St_n. We claim that there are a number k bounded by a function of D and a simple pattern P of a standard $(1/\psi)^k$-tiling such that k-refinement of P includes W.

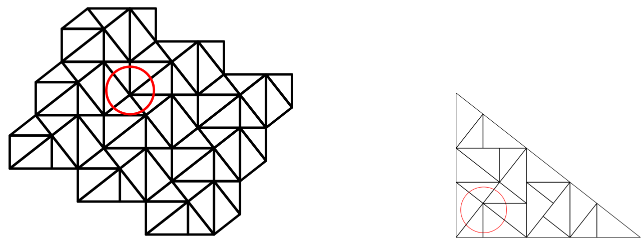
Let k be the minimal integer such that $\psi^k D$ is less that the constant ε from Lemma 1. If it happens that $k > n$ then let $k = n$. Let St_{n-k} denote the k-coarsening of the tiling St_n. If $k = n$ we are done, as we can let $P = T_{n-k}$. Otherwise, $\psi^k D < \varepsilon$ and hence the diameter of W measured in units $(1/\psi)^k$ is less than ε. By Lemma 1 W is covered by a simple pattern P of T_{n-k}. Hence the k-refinement of P includes W.

By Lemma 2 the number of simple patterns of T_{n-k} is bounded by a constant and k is bounded by a function of D. For each k and each simple pattern P the number of subsets of the k-refinement of P is finite. This completes the proof of the theorem modulo the lemmas. □

Theorem 1 gives a hope to describe LS tilings by a finite number of patterns. This would be possible if there were D such that every D-locally standard tiling is LS. The main result of this paper states that this is not the case.

Theorem 3. *For every* D *there is a* D*-locally standard tiling which is not locally standard and even not infinitely coarsenable.*

Proof. The proof of this theorem is fairly simple (but hard to find). Consider the following periodic 1-tiling U of the plane (the first configuration on the picture):



All its simple patterns are standard: they appear in the standard tiling of depth 6 (the second configuration on the picture).

Fix any D. Let ε be the constant from Lemma 1. Choose i so that $D\psi^i < \varepsilon$ and let U_i be the i-refinement of tiling U. Then U_i is the sought tiling.

Indeed, by Lemma 1 for every pattern of U_i of diameter less than D there is a simple pattern P of U such that W is a subset of the i-refinement of P. As we have seen, all simple patterns of U are standard and so does P. Hence W is a standard pattern as well.

On the other hand, being periodic, the tiling U_i is not infinitely coarsenable (actually, it admits only $i + 2$ coarsenings). $\square$

4 Acknowledgments

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A Proof of the Lemma 1

Assume that ε is small enough (in the end we will see how small it should be).

Let S denote the convex closure of W. Then U includes S. As convex closure has the same diameter as the set itself, the diameter of S is at most ε. Assuming that ε is less than the lengths of all sides of triangles from T we conclude that S has at most one node of a triangle from T. If it has such a node then let K be that (unique) node. In this case the simple pattern P of T with the center K covers S (and hence W). Indeed, if ε is less than the altitude of a small triangle then all the points of all the triangles from the simple pattern P are at the distance at least $\varepsilon > D$ from K. As the diameter of S is at most D, this implies all the points of S are at distance at most D from K and are thus covered by the pattern.

Assume now that S has no node of a triangle from T. If S covered by only one triangle we are done — any its node can be taken as the center of the sought simple pattern.

Otherwise S has no nodes of triangles from T and cannot be covered by one triangle from T. Let A be any triangle intersecting S, say in point C, and let D be any point from $S \setminus A$. Consider the segment $[C, D]$. At some point E that segment leaves the triangle A. The points of $[E, D]$ that lie very close to E belong to S and hence to some triangle B from T. That triangle includes the point E.

If E is close to a node of A or a node of B we can let K be that node. Indeed, all points in S are close to E and E is close to K. Hence all points from S are close to K and are thus covered by the simple pattern with center K. Otherwise E is the internal node of a leg of A and an internal node of a leg of B and thus A and B share a common segment. All nodes of A and B are far from E and hence from S. This implies that S is covered by $A \cup B$. It remains to notice that A and B belong to a simple pattern of T: indeed, both ends of the line segment shared by A, B can be chosen as the center of that simple pattern.

A calculation shows that ε equal to the half of altitude of the small triangle times ψ^2 will do.

B Proof of the Lemma 2

The second statement of the theorem (the number of simple patterns of standard d-tilings does not depend on d) is obvious. So we will assume that $d = 1$.

Consider standard 1-tilings of depth 9 and 10 (Fig. 1). A careful examination reveals that every simple pattern of the second tiling is isometric to a simple pattern of the first tiling. This implies that every simple pattern of standard 1-tiling of depth 11 is isometric to a simple pattern of the standard 1-tilings of depth 10.

Indeed, let K be a node of a triangle of St_{11} and P the simple pattern of St_{11} with center K. We claim that there is a node K' of a triangle from St_{10} such that the refinement of the simple pattern of St_{10} centered in K' includes the pattern P.

As K is a node of a triangle of St_{11}, it lies on a side a of a triangle from St_{10}. If K is a node of that triangle, then there is nothing to do: the refinement of the simple pattern of St_{10} centered in K obviously includes the pattern P.

Otherwise K' is the inner node on the side a. If K is on the border of the area tiled by St_{10} then the refinement of the simple pattern of St_{10} centered in any node of that triangle obviously includes the pattern P.

Otherwise K is also an inner node of a side of another triangle from St_{10}. Any two triangles that share an interval belong to a common simple pattern: as a center of such pattern we can take one of their nodes, namely, the node that is closest to the shared segment. The refinement of that pattern includes P.

As every simple pattern of St_{10} is isometric to a simple pattern of St_9, so is to the simple pattern of St_{10} centered in K'. Let U be a simple pattern of St_9 whom it is isometric. Let V stand for the image of K under that isomorphism. Then V is the center of a simple pattern of St_{10} that is isometric to P.

In the similar way we can show that every simple pattern of the standard 1-tiling of depth 12 is isometric to a simple pattern of the standard 1-tilings of depth 11 etc. Thus any simple pattern of any standard 1-tiling is isometric to a simple pattern of St_9.

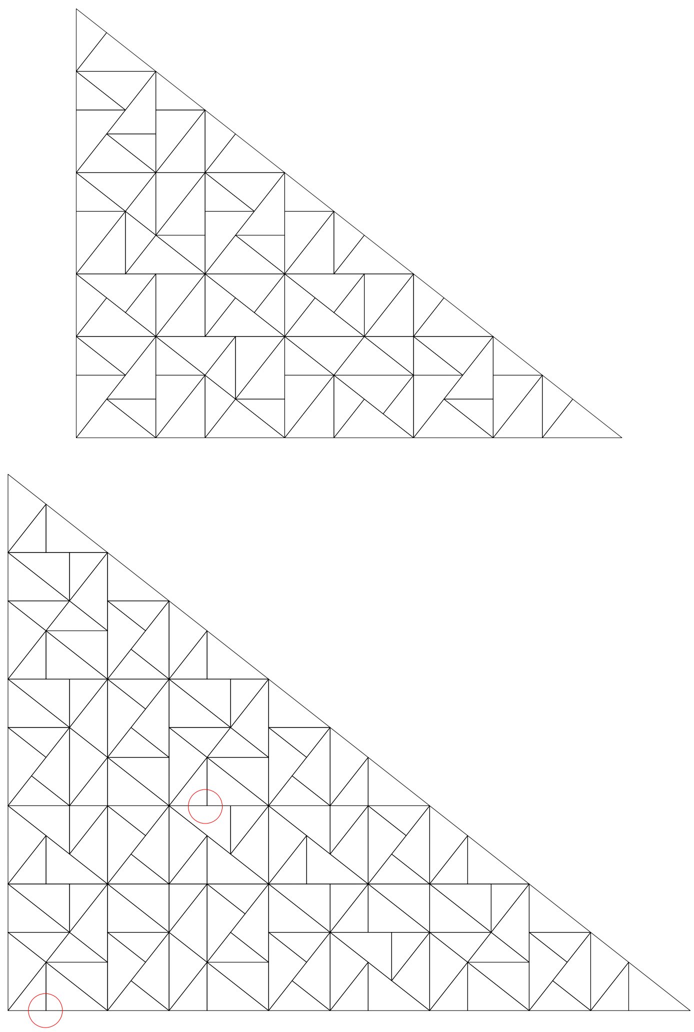

Figure 1: Standard tilings of depths 9 and 10