Logical operations and Kolmogorov complexity

Alexander Shen* Nikolai Vereshchagin†

Kolmogorov complexity $K(x)$ of a binary string x is defined as minimal length of a program that generates this string. This definition can be extended to sets of strings. Let A be a (finite or infinite) set of strings. We define the complexity $K(A)$ as the length of a shortest program that generates some string $x \in A$. Informally, we consider A as a problem "Generate any element of A"; $K(A)$ is complexity of this problem. Evidently, $K(A) = \min\{K(x) \mid x \in A\}$, so this generalization gives nothing really new.

However, it can be combined with the definition of logical operations on sets of strings that goes back to Kolmogorov's paper [3] and Kleene's notion of realizability [2]. Let A and B be two sets of strings. We define sets $A \wedge B$, $A \vee B$ and $A \to B$ as follows:

- $A \wedge B = \{\langle a, b\rangle \mid a \in A, b \in B\}$
- $A \vee B = \{\langle 0, a\rangle \mid a \in A\} \cup \{\langle 1, b\rangle \mid b \in B\}$
- $A \to B = \{p \mid [p](x) \in B \text{ for all } x \in A\}$

Here $\langle \cdot, \cdot \rangle$ is computable encoding of pair of strings; $[p](x)$ stands for the output of p (considered as a program) when applied to input x.

Example 1. Let x and y be two strings. Consider the set $x \to y$ (to simplify notation we identify a string s and the singleton $\{s\}$). This set contains all programs that map x to y. It is easy to see that $K(x \to y) = K(y|x) + O(1)$ where $K(y|x)$ denotes conditional complexity of the string y when x is known.

Note also that $K(x \wedge y)$ is the complexity of the pair (x, y) and $K(x \vee y) = \min(K(x), K(y))$.

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*Institute of Problems of Information Transmission (Moscow); Independent University of Moscow; e-mail: `shen@landau.ac.ru`, `shen@mccme.ru`.

†Dept. of Mathematical Logic and Theory of Algorithms, Moscow State University, Vorobjevy Gory, Moscow 119899, Russia; e-mail: `ver@mech.math.msu.su`, `ver@mccme.ru`.

Example 2. Now consider the set $x \leftrightarrow y$ defined as $(x \to y) \wedge (y \to x)$. By definition, $K(x \leftrightarrow y)$ is the complexity of pair of programs transforming x to y and vice versa. It turns out (as proved in [1]) that $K(x \leftrightarrow y) = \max(K(x|y), K(y|x)) + O(\log(K(x|y) + K(y|x)))$.

Example 3. Let x and y be two strings. Consider the set $(x \to y) \to y$. Its elements are programs that map every program in $(x \to y)$ (i.e., transforming x to y) to y. Let us prove that $K((x \to y) \to y) = \min(K(x), K(y)) + O(\log(|x| + |y|))$. (Here $|s|$ stands for the length of s.)

It is easy to see that $K((x \to y) \to y) \leq K(y) + O(1)$. Indeed, any program that prints y can be considered as a program that maps every element of $(x \to y)$ to y.

It is also easy to see that $K((x \to y) \to y) \leq K(x) + O(1)$. Indeed, for a given string x consider the program p_x that maps any program s to $[s](x)$. If s belongs to $x \to y$, then $[p_x](s) = [s](x) = y$. Therefore, $p_x \in ((x \to y) \to y)$. On the other hand, $K(p_x) \leq K(x) + O(1)$.

Therefore, $K((x \to y) \to y) \leq \min(K(x), K(y)) + O(1)$. It remains to prove that $\min(K(x), K(y)) \leq K((x \to y) \to y) + O(\log n)$ if x and y are strings of length at most n.

Let s be a program in $(x \to y) \to y$. Let S be the set of all strings of length at most n. For any function $\tau\colon S \to S$ fix some program l_τ that computes this function. A pair $(u, v) \in S \times S$ is called *s-coherent* if $[s](l_\tau) = v$ for any τ such that $\tau(u) = v$.

By definition the pair (x, y) is s-coherent. Other coherent pairs may exist. However, either all coherent pairs have x as the first component or all coherent pairs have y as the second component. To prove that, it is enough to prove the following statement: if (x', y') and (x'', y'') are s-coherent pairs then either $x' = x''$ or $y' = y''$. If it is not the case and $x' \neq x''$, $y \neq y''$, consider a function τ such that $\tau(x') = y'$ and $\tau(x'') = y''$. Then we have $[s](l_\tau) = y'$ and $[s](l_\tau) = y''$ at the same time, which is impossible, since $y' \neq y$.

If all s-coherent pairs have x as the first component, then $K(x) \leq K(s) + O(\log n)$, because we can find x when s and n are given (searching for a s-coherent pair and taking its first component). Similarly, if all s-coherent pairs have y as the second component, then $K(y) \leq K(s) + O(\log n)$. Therefore, $\min(K(x), K(y)) \leq K(s) + O(\log n)$. (End of the proof.)

Similar argument can be used to prove that $\min(K(x), K(z)) \leq K((x \to y) \to z) + O(\log n)$ for any strings x, y, z having length at most n. In particular, for $x = z$ we get $K((x \to y) \to x) = K(x) + O(\log n)$.

Example 4. For any three strings x, y, z having length at most n we have $K((x \vee y) \to z) = \max(K(z|x), K(z|y)) + O(\log n)$. This was recently proved by Andrei Muchnik with a very nice argument. One direction is easy: $K(z|x) = K(x \to z) \leq K((x \vee y) \to z)$; for the same reason $K(z|y) \leq K((x \vee y) \to z)$. To prove the reverse inequality, Muchnik assumes that $K(z|x) \leq k$ and $K(z|y) \leq k$ and proves that one can find a "fingerprint" f of z having length k such that z can be reconstructed from f and x and also from f and y. The proof uses expander graphs and will be published elsewhere.

Question. Let $A(p, q, \dots)$ be a propositional formula with connectivities $\wedge, \vee, \to$. For any strings $x, y, \dots$ consider the set $A(x, y \dots)$ defined in a natural way. The question is whether $K(A(x, y, \dots))$ is determined by complexities $K(x), K(y) \dots$ and conditional complexities of their combinations up to $O(\log n)$-term if $x, y, \dots$ are strings of length at most n. Examples 1–4 support this conjecture.

Remark 1. The goal of Kolmogorov [3] and Kleene [2] was to provide an interpretation of the intutionistic propositional calculus. Following this idea, one can prove the following statement: if $A(p, q, \dots)$ is provable in the intutionistic propositional calculus (IPC), then $K(A(x, y, \dots)) = O(1)$ for any strings $x, y, \dots$. Indeed, there exists a string s that belongs to $A(x, y, \dots)$ for all $x, y, \dots$ (induction by the length of the proof in IPC). See also [4] where a slightly different approach using Scott domains is used.

Remark 2. It is easy to see that

$$K(B) \leq K(A) + K(A \to B) + O(\log K(A \to B)))$$

Indeed, one can combine (self-delimiting encoding of) a program from $A \to B$ and (encoding of) any element of A to get an encoding of some element of B.

We can combine this remark with the previous one: if $A(p, q, \dots) \to B(p, q, \dots)$ is provable in IPC, then $K(B(x, y, \dots)) \leq K(A(x, y, \dots)) + O(1)$ for all strings $x, y, \dots$ For example, the formula $(x \vee y) \to ((x \to y) \to y))$ is provable in IPC, therefore $K((x \to y) \to y)) \leq K(x \vee y) + O(1) = \min(K(x), K(y)) + O(1)$ (as we mentioned in example 3 above).

Example 5. The Pierce law $((x \to y) \to x) \to x$ is not derivable in IPC. However, it has low complexity: $K(((x \to y) \to x) \to x) = O(\log n)$ for any strings x, y of length at most n. Indeed, let s belong to $(x \to y) \to x$. This time call a pair $(u, v) \in S \times S$ *s-coherent* if $[s](l_\tau) = u$ for any τ such that $\tau(u) = v$. If (u, v) is s-coherent then $u = x$, since otherwise there exists τ with $\tau(x) = y$, $\tau(u) = v$ and we have $x = [s](l_\tau) = u$. Given s find an s-coherent pair and output its first component. This instruction describes a program from $((x \to y) \to x) \to x$ of complexity $\log n$ (note that we need to know n).

Recalling Remark 2, we see again that $K(x) \leq K((x \to y) \to x) + O(\log n)$ (cf. Example 3, last sentence).

However, as the next example shows, the inequality for complexities may be valid even in the case when the corresponding implication has large complexity.

Example 6. $K(((x \to y) \to y) \to (x \vee y)) = \min(K(x|y), K(y|x)) + O(\log(|x| + |y|))$.

(Recall that $K((x \to y) \to y) = K(x \vee y) + O(\log n)$, as we have seen in Example 3, so one may expect that $K(((x \to y) \to y) \to (x \vee y)) = O(\log n)$. But this is not the case.)

The inequality $K(((x \to y) \to y) \to (x \vee y)) \leq K(y|x) + O(1)$ is straightforward since given a program p with $[p](x) = y$ and a program s in $(x \to y) \to y$ we can find $y = s[p]$.

Let us prove that $K(((x \to y) \to y) \to (x \vee y)) \leq K(x|y) + O(\log n)$, where $n = \max\{|x|, |y|\}$. It suffices to prove that given the triple $\langle n$, a program p with $[p](y) = x$, a program s in the set $(x \to y) \to y\rangle$ we are able to find either x or y. Recall the notion of s-coherent pair (see Example 3). Given n, p and s find an s-coherent pair (u, v). Then continue to enumerate other s-coherent pairs and run in parallel p on input v. We stop if we either find another s-coherent pair (u', v'), or we find out that $[p](v) = u$. In the former case we know either x, or y: if $v' \neq v$ then $x = u$ and if $u' \neq u$ then $y = v$ (recall that either the first component of all s-coherent pairs is equal to x, or the second component of all s-coherent pairs is equal to y). In the latter case (when $[p](v) = u$) we know that $x = u$. Indeed, if $x \neq u$ then $y = v$ hence $u = [p](v) = [p](y) = x$. Note that computation terminates (if there are no other s-coherent pairs except (u, v) then $(u, v) = (x, y)$ hence $[p](v) = u$).

Let us prove that $K(((x \to y) \to y) \to (x \vee y)) \geq \min\{K(y|x), K(x|y)\} - O(1)$. Assume that a program p is in the set $((x \to y) \to y) \to (x \vee y)$. We wish to prove that either given p we can find a program r_1 with $[r_1](x) = y$, or given p we can find a program r_2 with $[r_2](y) = x$.

We know that $p[s] = \langle 0, x\rangle$ or $p[s] = \langle 1, y\rangle$ for any s in $(x \to y) \to y$. Choose a pair A, B of enumerable inseparable subsets of $\mathbb{N}$. For any $i \in \mathbb{N}$

and any strings u, v consider the following program $q_i(u, v)$ in $(u \to v) \to v$: given a program s run it on input u and enumerate in parallel A and B; if it turns out that $[s](u) = v$ then output v and stop, if it turns out that $i \in A$ then output v and stop, if it turns out that $i \in B$ and $[s](u)$ is defined then output $[s](u)$ and stop (note that it does not matter which option to choose if several of them happen simultaneously). As for any i the program $q_i(x, y)$ is in $(x \to y) \to y$, the program p applied to $q_i(x, y)$ outputs either $\langle 0, x\rangle$ or $\langle 1, y\rangle$. And either there is $i \in A$ such that $[p](q_i(x, y)) = \langle 0, x\rangle$, or there is $i \in B$ such that $[p](q_i(x, y)) = \langle 1, y\rangle$ (otherwise the decidable set $\{i \mid [p](q_i(x, y)) = \langle 1, y\rangle\}$ separates A and B). In the former case given p and y we are able to find x: find $i \in A$ and u such that $[p](q_i(u, y)) = \langle 0, u\rangle$ and output u; note that $q_i(u, y)$ is in $(x \to y) \to y$ hence $u = x$. In the latter case given p and x we are able to find y: find $i \in B$ and v such that $[p](q_i(x, v)) = \langle 1, v\rangle$ and output v; note that $q_i(x, v)$ is in $(x \to y) \to y$ hence $v = y$. (End of proof.)

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