How to Use Expert Advice in the Case when Actual Values of Estimated Events Remain Unknown

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Abstract

The problem how to use experts whose perfomance is unknown has been studied in several recent papers. It has been assumed that after we have made a prediction we get know the actual value of the event.

In some situations, however, we never get know the actual outcomes. This is the case, for example, in diagnosing a disease, in some kinds of competitions, in making political decisions and so on.

In the present paper we consider a model in which we never get know actual outcomes. The predicted events are binary. Each expert independently on other experts estimates each outcome correctly with some probability depending on expert. His/her estimates of different events are mutually independent, too.

The performance of a strategy is measured as the probability of correct estimate on the worst-case outcome sequence. Our main result is the polynomial time strategy such that for any group of at least 3 experts the difference between its performance and the performance of the strategy being optimal for this expert group is $O(\sqrt{\log n/n})$, where n stands for the number of outcomes.

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1 Introduction

The problem how to use expert advice has been studied in several papers (see, for example [1, 3]). The common feature of the studied settings is that when predicting the current event we know the actual values of the previous events.

In some situations, however, we do not know the actual values of previous events while estimating current event.

Consider, for example, the following scenario. A sociologist performs a sociological research. He/she asks about 100 randomly chosen members of a political party trying to learn the "party's opinion" on about 100 questions. Each of the chosen members receives a questionnaire, fills it and returns to the sociologist. Each of them can be considered as an expert in the questions mentioned in the questionnaire. The sociologist have to process the filled questionnaires in order to learn the party's opinion on those questions.

The simplest idea is to estimate each item as as the majority of chosen members do. However this idea is not the best one. Indeed, it may happen that some of the asked members are much more competent than others. In this case it would better to estimate as the majority of most competent members. However, we do not know a priori which expert has the best performance.

In the present paper we consider the problem of using experts under the assumption that we do not get know the actual outcomes. This assumption forces to impose some restrictions on the experts because otherwise good strategies do not exist as we cannot distinguish between good and bad experts.

A natural requirement on experts is as follows: for any expert there exists a real number $p > 1/2$ such that on any sequence of events the performance (the frequency of correct estimates) of that expert is at least p. A more natural requirement is as follows: for any expert there is subinterval of interval $(1/2; 1)$ which contains his/her performance on any outcome sequence.

However, we admit for the sake of simplicity another restriction on experts, which is close to the second of the above restrictions but uses randomness. We assume that each expert is assigned a real (unknown) value in the segment $(1/2;1]$ called his/her *competence.* Every expert gives the correct value of each outcome with probability being equal to his/her competence. Every expert acts independently on other experts and his/her estimates of different outcomes are mutually independent, too. Given the estimates of all the experts of outcomes number $1, 2, \dots, n$ we have to estimate the last (nth outcome).[1] In this form, the problem was set up by Andrey Muchnik (personal conversation).

How to measure the performance of a strategy? Of course, the performance of a strategy may depend on the actual outcome sequence. For example, the strategy "return always 1" is good if all actual outcomes are equal to 1 and is bad if all actual outcomes are equal to 0. Likewise, the performance of a strategy may depend on experts. For example, the strategy "estimate as the first expert" is good if the first expert has high competence and is bad if the first expert has low competence. We measure the performance of a strategy *on* an outcome sequence *with respect to* an expert group as the probability of the correct estimate.

The goal is to find a strategy whose performance is high on *any* outcome sequence with respect to *any* expert group. Thus we admit here the analysis in the worst case: we make no assumptions about actual outcome sequence.[2]

It is easy to see that no strategy can have high worst-case performance with respect to an expert group in which all experts have competence close to 1/2. Therefore it is natural to compare the worst-case performance of a strategy (with respect to an expert group) with the worst-case performance of the strategy which is optimal with respect to this particular group. The strategy $\tilde{E}_G$ being optimal with respect to given group G is the well known maximum likelihood estimator. To estimate an outcome, $\tilde{E}_G$ takes estimates $e_1, e_2, \dots, e_m$ of all the experts of that outcome and returns the $v \in \{0,1\}$ for which the probability of event 'for all $i \leq m$ the ith expert in G gives estimate e_i provided the actual outcome is v' is greater. Actually, the performance of the maximal likelihood estimator does not depend on the outcome sequence.

So we will compare the performance of a strategy with respect to an expert group G with the performance of $\tilde{E}_G$. Our main result is the polynomial time strategy S whose worst-case performance is asymptotically equal (when the length of outcome sequence goes to infinity) to the performance of $\tilde{E}_G$ *for any group G having at least 3 experts* (Theorem 1). Actually, the performance of S does not depend on the outcome sequence. More precisely, Theorem 1 states that the difference between the performance of S and the performance of $\tilde{E}_G$ is $O(\sqrt{\log n/n})$ with respect to any group G having at least 3 experts (the constant hidden in "O notation" depend on G). Thus for large n the expected frequency of correct estimates of S is approximately equal to the performance of $\tilde{E}_G$.

So, for example, if there are three experts of competence $p_1 \geq p_2 \geq p_3 > 1/2$ then the strategy will correctly predict nth outcome with probability at least $\Theta(p_1,p_2,p_3) - O(\sqrt{\log n/n})$, where

$$\begin{aligned} &\Theta(p_1,p_2,p_3) \\ &\quad = \quad p_1p_2p_3 + p_1p_2(1-p_3) + p_1(1-p_2)p_3 \\ &\quad + \quad \min\{p_1(1-p_2)(1-p_3), (1-p_1)p_2p_3\}. \end{aligned}$$

It is easy to see that if $p_1(1-p_2)(1-p_3) \geq (1-p_1)p_2p_3$ then $\Theta(p_1,p_2,p_3) = p_1$. Otherwise $\Theta(p_1,p_2,p_3)$ is equal to the performance of the strategy "estimate as the majority of experts".

The considered framework is close to the framework studied in the paper [4], where it is shown that classifications based on m conditional independent Boolean features can efficiently be learned by examples.

In Section 4 we consider an alternate way to define performance of strategies. We consider there the uniform probability distribution over outcome sequences and the uniform probability distribution over competence of experts (that is, over $(1/2;1]^m$). The performance of a strategy is defined as the expected probability of correct estimate on a random outcome sequence with respect to a random expert group. For this way of measuring performance there exists an optimal strategy which can be run in polynomial time if the number of experts or the number of outcomes is fixed.

In Section 5 we investigate some properties of the performance of maximum likelihood estimator, that performance reflecting the power of expert systems. In particular, for any expert group we find how much should be the competence of an expert in order that his/her including in that group increases its power.

Notation We denote $1-x$ by $\bar{x}$.

2 The Maximum Likelihood Estimator

In this section we assume that there are m experts having competence respectively $p_1, p_2, \dots, p_m > 1/2$, which *are known.* So to estimate the current outcome we do

[1]To estimate other outcomes we may apply the same strategy.

[2]But one exception: in Section 4 we consider the uniform measure on outcome sequences.

not need estimates of previous outcomes. Therefore, in this section, a strategy is a function from $\{0,1\}^m$ into $\{0,1\}$. To distinguish between strategies in the sense of the next section, which are functions from $\{0;1\}^{mn}$ into $\{0,1\}$, we will call functions from $\{0,1\}^m$ into $\{0,1\}$ *estimators*.

In the sequel we denote by $\mathbf{p}$ the sequence $p_1, p_2, \dots, p_m$. Let θ_1^v, θ_2^v, ..., θ_m^v, $v = 0, 1$, be mutually independent random values such that $\mathrm{P}\{\theta_i^v = v\} = p_i$. The random value θ_i^v is the estimate of expert i of the unknown outcome provided the actual outcome is v.

Definition 1 The performance $Q(\mathbf{p}, E)$ of an estimator E with respect to the group of experts having competence $\mathbf{p}$ is the the minimal of two numbers

$$\mathrm{P}[E(\theta_1^0, \theta_2^0, \dots, \theta_m^0) = 0], \qquad \mathrm{P}[E(\theta_1^1, \theta_2^1, \dots, \theta_m^1) = 1].$$

Let us define now the maximum likelihood estimator $\tilde{E}$:

$$\tilde{E}(e_1, e_2, \dots, e_m) = \begin{cases} 0 & \text{if } \mathrm{P}[\theta_1^0 = e_1, \dots, \theta_m^0 = e_m] \\ & \quad > \mathrm{P}[\theta_1^1 = e_1, \dots, \theta_m^1 = e_m], \\ 1 & \text{if } \mathrm{P}[\theta_1^0 = e_1, \dots, \theta_m^0 = e_m] \\ & \quad < \mathrm{P}[\theta_1^1 = e_1, \dots, \theta_m^1 = e_m], \\ e_1 & \text{otherwise} \end{cases}$$

Of course the maximum likelihood estimator depends on the sequence $\mathbf{p} = p_1, \dots, p_m$. When this sequence is not clear from the context we will write $\tilde{E}_{\mathbf{p}}$ instead of $\tilde{E}$.

The next lemma, which is proven in Appendix, is well known; it states the optimality of the estimator $\tilde{E}$.

Lemma 1 *$Q(\mathbf{p}, \tilde{E}_{\mathbf{p}}) \geq Q(\mathbf{p}, E)$ for all E and all $\mathbf{p}$.*

Example 1 Let us be given a group of three experts with competence $p_1 \geq p_2 \geq p_3 > 1/2$. Then $\tilde{E}(000) = \tilde{E}(001) = \tilde{E}(010) = 0$, $\tilde{E}(111) = \tilde{E}(110) = \tilde{E}(101) = 1$,

$$\tilde{E}(011) = \begin{cases} 0, & \text{if } p_1\bar{p}_2\bar{p}_3 \geq \bar{p}_1 p_2 p_3, \\ 1, & \text{else,} \end{cases}$$

and $\tilde{E}(100) = \overline{\tilde{E}(011)}$.

So the optimal estimator estimates as the first expert if $p_1\bar{p}_2\bar{p}_3 \geq \bar{p}_1 p_2 p_3$ and estimates as the majority of experts otherwise. Its performance is equal to p_1 in the first case and to $p_1p_2p_3 + \bar{p}_1p_2p_3 + p_1\bar{p}_2p_3 + p_1p_2\bar{p}_3$ in the second case.

3 Asymptotically Optimal Strategy

In this section we assume that there are $m \geq 3$ experts having competence respectively $p_1, p_2, \dots, p_m > 1/2$, which *are unknown.* Those experts estimate n binary outcomes $v_1, v_2, \dots, v_n$, which are unknown, too. Given all those estimates we have to estimate the last (nth) outcome. So a *m, n-strategy* is a function from the set $(\{0,1\}^m)^n$ into $\{0,1\}$. The family of m, n-strategies is denoted by $\mathcal{S}_{mn}$. A *strategy* is a family $S = \{S_{mn} \mid m, n = 1, 2, \dots\}$ of functions such that S_{mn} is a m, n-strategy for any m, n.

The estimate of expert i of jth outcome is a random value depending on the actual outcome v_j. This estimate is equal to v_j with probability p_i. The estimates of different experts are mutually independent and the estimates of the same expert of different outcomes are mutually independent, too.

Let us define the *performance $Q(\mathbf{p}, S)$ of a m, n-strategy S with respect to experts with competence* $\mathbf{p} = p_1, p_2, \dots, p_m$. Let ξ_{ij}^v stand for the estimate of expert i of the jth outcome provided its actual value is v. Then

$$Q(\mathbf{p}, S) = \min_{v_1, \dots, v_n} \mathrm{P}[S(\xi_{11}^{v_1}\xi_{21}^{v_1} \cdots \xi_{m1}^{v_1} \cdots \xi_{1n}^{v_n}\xi_{2n}^{v_n} \cdots \xi_{mn}^{v_n}) = v_n].$$

It is easy to see that $Q(\mathbf{p}, S) \leq Q(\mathbf{p}, \tilde{E}_{\mathbf{p}})$ for any m, n-strategy S and for any $\mathbf{p}$.

Theorem 1 *There exists a strategy S computable in polynomial time (in m, n) such that*

$$Q(\mathbf{p}, S_{mn}) \geq Q(\mathbf{p}, \tilde{E}_{\mathbf{p}}) - O(\sqrt{\log n/n})$$

for any $m \geq 3$ and any $\mathbf{p}$ (the constant hidden in $O(\sqrt{\log n/n})$ depends on m and on $\mathbf{p}$).

Before to prove the theorem let us remind the Chernoff bound, which will be used in the proof.

Theorem 2 *(Chernoff bound [2]) Let $\xi_1, \dots, \xi_n$ be independent random values in the set $\{0,1\}$ such that $\mathrm{P}[\xi_i = 1] = p$ for all i. Then for any $\gamma \in [0; p(1-p)]$,*

$$\mathrm{P}\left[\left|\frac{1}{n}\sum_{i=1}^{n} \xi_i - p\right| \geq \gamma\right] \leq 2e^{-2\gamma^2 n}.$$

Proof of Theorem 1. The m, n-strategy S_{mn} is computed in two stages.

First stage: given estimates of all the experts of outcomes number $1, 2, \dots, n-1$ compute the approximate competence $\hat{p}_1, \dots, \hat{p}_m$ of experts.

Second stage: given $\hat{\mathbf{p}} = \hat{p}_1, \dots, \hat{p}_m$ and the estimates of the last outcome apply the maximum likelihood estimator substituting $\hat{p}_1, \dots, \hat{p}_m$ for unknown $p_1, \dots, p_m$.

Let us turn to the first stage.

First stage. Consider first the case $m = 3$. Let r_1, r_2, r_3 be defined by equalities $p_1 = 1/2 + r_1$, $p_2 = 1/2 + r_2$,

$p_3 = 1/2 + r_3$. Then r_1, r_2, r_3 are greater than 0. Let s_{12} stand for the probability that the estimates of the first and the second experts (of the same outcome) coincide. It is easy to see that $s_{12} = 1/2 + 2r_1 r_2$. The crucial point is that this probability does not depend on the actual outcome. In the same way define s_{13} and s_{23}. Note that given s_{12}, s_{13} and s_{23} we can find r_1, r_2 and r_3 by using the simple formulae:

$$r_1 = \sqrt{\tfrac{(2s_{12}-1)(2s_{13}-1)}{4(2s_{23}-1)}}, \quad r_2 = \sqrt{\tfrac{(2s_{12}-1)(2s_{23}-1)}{4(2s_{13}-1)}},$$
$$r_3 = \sqrt{\tfrac{(2s_{13}-1)(2s_{23}-1)}{4(2s_{12}-1)}}.$$

This observation leads to the following algorithm for computing approximations $\hat{p}_1$, $\hat{p}_2$, $\hat{p}_3$ to competence of experts. Let e_{11}, e_{21}, e_{31}, e_{12}, e_{22}, e_{32}, ..., e_{1n}, e_{2n}, e_{3n} be the estimates given by experts. Find first the value $\hat{s}_{12} \rightleftharpoons |\{j < n | e_{1j} = e_{2j}\}|/(n-1)$. Find $\hat{s}_{13}$ and $\hat{s}_{23}$ defined in the same way. If some of the numbers $2\hat{s}_{12} - 1, 2\hat{s}_{13} - 1, 2\hat{s}_{23} - 1$ is not positive let $\hat{p}_1 = \hat{p}_2 = \hat{p}_3 = 1$. Otherwise substitute in the three above formulae $\hat{s}_{12}$, $\hat{s}_{13}$, $\hat{s}_{23}$ respectively for s_{12}, s_{13}, s_{23} and denote the resulting values by $\check{r}_1$, $\check{r}_2$, $\check{r}_3$. Find rational approximations $\hat{r}_1$, $\hat{r}_2$, $\hat{r}_3$ respectively to $\check{r}_1$, $\check{r}_2$, $\check{r}_3$ with precision say $1/n$. Let $\hat{p}_1 = 1/2 + \hat{r}_1$, $\hat{p}_2 = 1/2 + \hat{r}_2$ and $\hat{p}_3 = 1/2 + \hat{r}_3$.

The case $m = 3$ is done. In the case $m > 3$ we find $\hat{p}_1$, $\hat{p}_2$, ..., $\hat{p}_m$ as follows: $\hat{p}_1$, $\hat{p}_2$, $\hat{p}_3$ are computed as earlier and to find $\hat{p}_i$ for $i > 3$ we consider the group consisting of the first, the second and the ith expert and do the same thing. The first stage is completed.

On the second stage we return the value $\tilde{E}_{\hat{\mathbf{p}}}(e_{1n}, e_{2n}, \ldots, e_{mn})$, where $\hat{\mathbf{p}} = \hat{p}_1, \hat{p}_2, \ldots, \hat{p}_m$.

The strategy S_{mn} is defined. Let us prove that for any $\mathbf{p}$,

$$Q(\mathbf{p}, S_{mn}) \geq Q(\mathbf{p}, \tilde{E}_{\mathbf{p}}) - O(\sqrt{\log n/n}).$$

It is easy to see that, for the constructed strategy, the probability of correct estimate does not depend on the outcome sequence. So we will estimate this probability assuming that all outcomes are equal to 1.

Let us fix arbitrary $p_1, p_2, \ldots, p_m > 1/2$. Denote $\min\{p_1 - 1/2, p_2 - 1/2, \ldots, p_m - 1/2\}$ by ε. Then we have $s_{jk} \geq 1/2 + 2\varepsilon^2$ for all $j < k \leq m$.

Let $\gamma = \sqrt{\ln n/n}$. Let us estimate the probability of event $|\hat{s}_{jk} - s_{jk}| \leq \gamma$ using Chernoff bound. The conditions of Theorem 2 are fulfilled for large enough n unless $s_{jk} = 1$. In the case $s_{jk} = 1$ that probability is equal to 1. By Chernoff inequality any of the events

$$|\hat{s}_{jk} - s_{jk}| \leq \gamma \tag{1}$$

for $j < k \leq m$ holds with probability at least $1 - 2e^{-2\ln n} = 1 - 2n^{-2}$.

Therefore we have

$$Q(\mathbf{p}, S_{mn}) = \mathrm{P}[S_{mn}(\xi^1_{11}\xi^1_{21}\ldots\xi^1_{mn}) = 1] \geq$$
$$\mathrm{P}[\tilde{E}_{\hat{\mathbf{p}}}(\xi^1_{1n}, \ldots \xi^1_{mn}) = 1 \mid \forall j < k \leq m \ |\hat{s}_{jk} - s_{jk}| \leq \gamma]$$
$$-m^2 n^{-2}.$$

Let us note that for any fixed estimator E the events $E(\xi^1_{1n}, \ldots \xi^1_{mn}) = 1$ and $\forall j < k \leq m \ |\hat{s}_{jk} - s_{jk}| \leq \gamma$ are independent. The inequalities (1) imply that the estimator $\tilde{E}_{\hat{\mathbf{p}}}$ is close to estimator $\tilde{E}_{\mathbf{p}}$. The value $Q(\mathbf{p}, \tilde{E}_{\hat{\mathbf{p}}})$ is continuous when $\hat{\mathbf{p}}$ varies. Therefore,

$$\mathrm{P}[\tilde{E}_{\hat{\mathbf{p}}}(\xi^1_{1n}, \ldots \xi^1_{mn}) = 1 \mid \forall j < k \leq m \ |\hat{s}_{jk} - s_{jk}| \leq \gamma]$$

is close to $Q(\mathbf{p}, E_{\mathbf{p}})$. The exact bound follows from the next two lemmas.

Lemma 2 *If n is large enough and (1) is true for all $j < k \leq m$ then $|\hat{p}_j - p_j| < \varepsilon^{-5}\gamma + 1/n$ for any $j \leq m$.*

Proof. Let n be so large that $\gamma < 1.5\varepsilon^2$ and let (1) be true for all $j < k \leq m$. Let us prove that $|\hat{p}_1 - p_1| < \varepsilon^{-5}\gamma + 1/n$. We have $2\hat{s}_{12} - 1 \geq \varepsilon^2$, $2\hat{s}_{13} - 1 \geq \varepsilon^2$ and $2\hat{s}_{23} - 1 \geq \varepsilon^2$. Therefore

$$\begin{aligned}
|p_1 - \hat{p}_1| &= |r_1 - \hat{r}_1| \leq |r_1 - \check{r}_1| + 1/n \\
&= \left|\sqrt{\frac{(2s_{12}-1)(2s_{13}-1)}{4(2s_{23}-1)}} - \sqrt{\frac{(2\hat{s}_{12}-1)(2\hat{s}_{13}-1)}{4(2\hat{s}_{23}-1)}}\right| \\
&\quad + \frac{1}{n} \\
&= \Big(\Big|(2s_{12}-1)(2s_{13}-1)(2\hat{s}_{23}-1) \\
&\quad -(2\hat{s}_{12}-1)(2\hat{s}_{13}-1)(2s_{23}-1)\Big| \\
&\Big/ \quad 2\Big(\sqrt{(2s_{12}-1)(2s_{13}-1)(2s_{23}-1)(2\hat{s}_{23}-1)^2} \\
&\quad + \sqrt{(2\hat{s}_{12}-1)(2\hat{s}_{13}-1)(2\hat{s}_{23}-1)(2s_{23}-1)^2}\Big)\Big) + \frac{1}{n} \\
&\leq \frac{3 \cdot 4 \cdot 2\gamma}{2(\sqrt{256\varepsilon^{10}} + \sqrt{128\varepsilon^{10}})} + \frac{1}{n} \leq \frac{\gamma}{\varepsilon^5} + \frac{1}{n}. \ \Box
\end{aligned}$$

Lemma 3 *Let θ_1, ..., θ_m be independent random values such that $\mathrm{P}[\theta_i = 1] = p_i$. Then $\mathrm{P}[\tilde{E}_{\hat{\mathbf{p}}}(\theta^1_1, \ldots, \theta^1_m) = 1] = Q(\mathbf{p}, \tilde{E}_{\hat{\mathbf{p}}})$ and if $|\hat{p}_i - p_i| \leq \delta$ for all $i \leq m$ then*

$$|\, Q(\mathbf{p}, \tilde{E}_{\hat{\mathbf{p}}}) - Q(\mathbf{p}, \tilde{E}_{\mathbf{p}})| < m2^m\delta.$$

This lemma will be proven in the Appendix.

Letting in the above lemma $\delta = \gamma/\varepsilon^5 + 1/n$ we get

$$\begin{aligned}
Q(\mathbf{p}, S_{mn}) &= \mathrm{P}[S_{mn}(\xi^1_{11}\xi^1_{21}\ldots\xi^1_{mn}) = v_n] \\
&\geq Q(\mathbf{p}, \tilde{E}_{\mathbf{p}}) - m2^m(\gamma/\varepsilon^5 + 1/n) - m^2 n^{-2} \\
&= Q(\mathbf{p}, \tilde{E}_{\mathbf{p}}) - O(\sqrt{\log n/n}). \ \Box
\end{aligned}$$

Remark 1. As we can see from the proof, the gap between the performance of the strategy's estimator and that of the ML estimator is exponential in the number of experts. So the fact that it is polynomial time in the number of experts is not so useful, since number of outcomes needs to be exponentially large in the number of experts for a given desired relative performance.

4 The Optimal Strategy for an Alternate Way to Define Performance of Strategies

In this section we will measure the performance of strategies as follows: for a m,n-strategy S let

$$\check{Q}(S) = \int \Big\{ 1/2^n \sum_{v_1,\ldots,v_n} \mathrm{P}[S(\xi_{11}^{v_1}\cdots\xi_{mn}^{v_n}) = v_n] \Big\} dp_1\cdots dp_m,$$

where the integral is taken over $(1/2, 1]^m$.

A m,n-strategy S is called *optimal* if $\check{Q}(S) \geq \check{Q}(S')$ for all m,n-strategies S'. A strategy $S = \{S_{mn} \mid m,n = 0,1,2,\ldots\}$ is called *optimal* if S_{mn} is optimal for all m,n.

Theorem 3 *There is an optimal strategy computable in polynomial time if n is fixed or m is fixed.*

Proof. Let $\check{Q}_{mn} = \max_{S\in\mathcal{S}_{mn}} \check{Q}(S)$. By definition we have

$$\check{Q}_{mn} = \max_{S\in\mathcal{S}_{mn}} \int 2^{-n} \sum_{\mathbf{v}} \mathrm{P}\left[S(\xi_{11}^{v_1},\ldots,\xi_{mn}^{v_n}) = v_n\right] dp_1\cdots dp_m.$$

Obviously,

$$\begin{aligned}\mathrm{P}[S(\xi_{11}^{v_1},\ldots,\xi_{mn}^{v_n}) = v_n] &= \sum_{e_{11},\ldots,e_{mn}} \mathrm{P}[S(e_{11},\ldots,e_{mn}) = v_n \wedge \\ &\qquad \forall i\leq m \forall j \leq n\ \xi_{ij}^{v_j} = e_{ij}].\end{aligned}$$

Therefore,

$$\begin{aligned}&\int 2^{-n}\sum_{\mathbf{v}}\sum_{e_{11},\ldots,e_{mn}} \mathrm{P}[S(e_{11},\ldots,e_{mn}) = v_n \wedge \\ &\qquad \forall i\leq m\forall j\leq n\ \xi_{ij}^{v_j} = e_{ij}] dp_1\ldots dp_m \\ &= \sum_{e_{11},\ldots,e_{mn}} \int 2^{-n}\sum_{\mathbf{v}} \mathrm{P}[S(e_{11},\ldots,e_{mn}) = v_n \wedge \\ &\qquad \forall ij\ \xi_{ij}^{v_j} = e_{ij}] dp_1\cdots dp_m.\end{aligned}$$

Given $e\in\{0,1\}$ denote by $F(e,e_{11},\ldots,e_{mn})$ the value

$$\int 2^{-n}\sum_{\mathbf{v}} \mathrm{P}[e = v_n \wedge \forall ij\ \xi_{ij}^{v_j} = e_{ij}] dp_1\cdots dp_m. \tag{2}$$

Then

$$\begin{aligned}\check{Q}_{mn} &= \max_S \sum_{e_{11},\ldots,e_{mn}} F(S(e_{11},\ldots,e_{mn}), e_{11},\ldots,e_{mn}) \\ &= \sum_{e_{11},\ldots,e_{mn}} \max_{e\in\{0,1\}} F(e,e_{11},\ldots,e_{mn}).\end{aligned}$$

Therefore the strategy that given a tuple $\langle e_{11},\ldots,e_{mn}\rangle$ returns 0 if

$$F(0,e_{11},\ldots,e_{mn}) > F(1,e_{11},\ldots,e_{mn})$$

and 1 otherwise is optimal. Note that this is again the maximum likelihood estimator. Indeed, $F(e,e_{11},\ldots,e_{mn})$ is equal to the probability that the last outcome in a random outcome sequence $\mathbf{v}$ is equal to e and random experts give estimates $e_{11},\ldots,e_{mn}$ on $\mathbf{v}$.

Let us prove that this strategy is polynomial time if either m or n is fixed.

Assume first that n is fixed. It suffices to prove that given e and $\langle e_{11},\ldots,e_{mn}\rangle$ we can compute in polynomial time $F(e,e_{11},\ldots,e_{mn})$. We have

$$\begin{aligned}&F(e,e_{11},\ldots,e_{mn}) \\ &= \int \Big(2^{-n}\sum_{\mathbf{v}} \mathrm{P}[e = v_n \wedge \forall ij \xi_{ij}^{v_j} = e_{ij}]\Big) dp_1\cdots dp_m \\ &= \int \Big(2^{-n}\sum_{\mathbf{v}:v_n=e} \mathrm{P}[\forall ij\ \xi_{ij}^{v_j} = e_{ij}]\Big) dp_1\cdots dp_m \\ &= \int \Big(2^{-n}\sum_{\mathbf{v}:v_n=e}\prod_{ij} \mathrm{P}[\xi_{ij}^{v_j} = e_{ij}]\Big) dp_1\cdots dp_m \\ &= \sum_{\mathbf{v}:v_n=e} 2^{-n}\int\Big(\prod_{ij}\mathrm{P}[\xi_{ij}^{v_j} = e_{ij}]\Big) dp_1\cdots dp_m.\end{aligned}$$

As the number of $\mathbf{v}$'s is 2^{n-1} (hence does not depend on m), it suffices to compute in polynomial time given $\mathbf{v}$ the value $\int(\prod_{ij}\mathrm{P}[\xi_{ij}^{v_j} = e_{ij}])dp_1\ldots dp_m$. This expression can be rewritten as $\prod_{i=1}^m(\int\prod_{j=1}^n \mathrm{P}[\xi_{ij}^{v_j} = e_{ij}]dp_i)$. The value

$$\mathrm{P}[\xi_{ij}^{v_j} = e_{ij}] = \begin{cases} p_i & \text{if } v_j = e_{ij}, \\ 1-p_i & \text{otherwise}\end{cases}$$

is a linear polynomial in p_i with coefficients not exceeding 1 in absolute value. By multiplying $\mathrm{P}[\xi_{ij}^{v_j} = e_{ij}]$ for $j = 1,2,\ldots,n$ we find in time not depending on m the coefficients of the polynomial $g_i(p_i) = \prod_{j=1}^n \mathrm{P}[\xi_{ij}^{v_j} = e_{ij}]$. Those coefficients do not exceed 2^n in absolute value. Then we find $\int g_i(p_i)dp_i$ and multiply the resulting values for $i = 1,2,\ldots,m$.

Assume now that m is fixed. Again it suffices to prove that given e and $\langle e_{11},\ldots,e_{mn}\rangle$ we can compute in polynomial time $F(e,e_{11},\ldots,e_{mn})$. In this case we rewrite $F(e,e_{11},\ldots,e_{mn})$ as

$$F(e,e_{11},\ldots,e_{mn})$$

$$= \int \Big(2^{-n} \sum_{\mathbf{v}:v_n=e} \mathrm{P}[\forall_{ij} \xi_{ij}^{v_j} = e_{ij}]\Big) dp_1 \cdots dp_m$$

$$= \int \Big(2^{-n} \sum_{\mathbf{v}:v_n=e} \prod_j \prod_i \mathrm{P}[\xi_{ij}^{v_j} = e_{ij}]\Big) dp_1 \cdots dp_m$$

Let us note that

$$\sum_{\mathbf{v}:v_n=e} \prod_j \prod_i \mathrm{P}[\xi_{ij}^{v_j} = e_{ij}]$$

$$= \prod_{j=1}^{n-1} \Big\{ \Big(\prod_{i=1}^{m} \mathrm{P}[\xi_{ij}^{0} = e_{ij}] + \prod_{i=1}^{m} \mathrm{P}[\xi_{ij}^{1} = e_{ij}] \Big) \times \prod_{i=1}^{m} \mathrm{P}[\xi_{ij}^{e} = e_{nj}] \Big\}.$$

Therefore,

$$F(e, e_{11}, \ldots, e_{mn})$$

$$= \int 2^{-n} \prod_{j=1}^{n-1} \Big\{ \Big(\prod_{i=1}^{m} \mathrm{P}[\xi_{ij}^{0} = e_{ij}] + \prod_{i=1}^{m} \mathrm{P}[\xi_{ij}^{1} = e_{ij}] \Big) \times \prod_{i=1}^{m} \mathrm{P}[\xi_{ij}^{e} = e_{nj}] \Big\} dp_1 \cdots dp_m.$$

The expressions $f_j(p_1, p_2, \ldots, p_m) = \prod_{i=1}^{m} \mathrm{P}[\xi_{ij}^{0} = e_{ij}] + \prod_{i=1}^{m} \mathrm{P}[\xi_{ij}^{1} = e_{ij}]$ for $j = 1, 2, \ldots, n-1$ and the expression $f_n(p_1, p_2, \ldots, p_m) = \prod_{i=1}^{m} \mathrm{P}[\xi_{in}^{e} = e_{in}]$ are multilinear polynomials (i.e., of degree 1 in every variable) in $p_1, \ldots, p_m$ and have at most 2^m integer coefficients not exceeding 2^{m+1} in absolute value. In time depending only on m we can find the coefficients of any of those polynomials. Therefore, in time linear in n we can find the coefficients of all those polynomials.

Now we have to multiply those n multilinear polynomials. Note that we cannot do that directly because direct multiplying of n polynomials having 2^m terms yields $(2^m)^n$ terms. To avoid this difficulty let us note that the resulting polynomial $f = f_1 f_2 \cdots f_n$ has degree n in every variable and therefore has $(n+1)^m$ coefficients. Therefore we can compute the coefficients of polynomials $f_1 f_2$, $f_1 f_2 f_3$, …, $f_1 f_2 \cdots f_n$ in succession. On every step we multiply two polynomials of degree at most n. This multiplying requires only $(n+1)^{2m}$ arithmetical operations.

Then we can find the integral of that polynomial by integrating separately all its terms. □

5 Comparing different expert groups

The value $\Theta(p_1, p_2, \ldots, p_m) \rightleftharpoons Q(p_1, p_2, \ldots, p_m, \tilde{E})$ reflects the competence of the expert group and will be called *the competence of the group* $p_1, p_2, \ldots, p_m$.

Theorem 4 *The function* $\Theta(p_1, p_2, \ldots, p_m)$ *is monotone, i.e.,* $p_1 \geq p_1', \ldots, p_m \geq p_m' \Rightarrow \Theta(p_1, \ldots, p_m) \geq \Theta(p_1', \ldots, p_m')$.

We will prove this theorem later.

The next theorem answers the following question: suppose we want to increase the competence of an expert group by including a new expert. How high should be his/her competence?

Let $x^{(1)}$ denote $1 - x$ and $x^{(0)}$ denote x. Let $\psi(y, z) = \frac{\max\{y,z\}}{y+z}$ and

$$t = \min_{e_1, \ldots, e_{m-1}} \psi(p_1^{(e_1)} p_2^{(e_2)} \cdots p_{m-1}^{(e_{m-1})}, p_1^{(\bar{e}_1)} p_2^{(\bar{e}_2)} \cdots p_{m-1}^{(\bar{e}_{m-1})}).$$

Theorem 5 $\Theta(p_1, \ldots, p_{m-1}, p_m) > \Theta(p_1, \ldots, p_{m-1})$ *for any* $p_m > t$ *and* $\Theta(p_1, \ldots, p_{m-1}, p_m) = \Theta(p_1, \ldots, p_{m-1})$ *for any* $1/2 \leq p_m \leq t$.

For example, assume that $p_i = p$ for $i = 1, 2, \ldots, m-1$. Then it is easy to see that $t = 1/2$ if $m-1$ is even and $t = p$ otherwise. Thus if $m-1$ is even including of any expert will improve the expert group and if $m-1$ is odd only including of experts with competence higher than p can improve the group.

Proof of Theorem 4 and Theorem 5. To prove the former theorem it suffices to prove that the function $\Theta(p_1, p_2, \ldots, p_m)$ is monotone in p_m.

It is easy to see that

$$\Theta(p_1, \ldots, p_m) = 0.5 \sum_{e_1, \ldots, e_m} \max\{p_1^{(e_1)} \cdots p_m^{(e_m)}, p_1^{(\bar{e}_1)} \cdots p_m^{(\bar{e}_m)}\}$$

$$= 0.5 \sum_{e_1, \ldots, e_{m-1}} \Big(\max\{p_1^{(e_1)} \cdots p_{m-1}^{(e_{m-1})} p_m, p_1^{(\bar{e}_1)} \cdots p_{m-1}^{(\bar{e}_{m-1})} (1 - p_m)\} + \max\{p_1^{(e_1)} \cdots p_{m-1}^{(e_{m-1})} (1 - p_m), p_1^{(\bar{e}_1)} \cdots p_{m-1}^{(\bar{e}_{m-1})} p_m\} \Big)$$

For any positive α, β the function $f_{\alpha\beta}(x) = \max\{\alpha x, \beta(1-x)\} + \max\{\alpha(1-x), \beta x\}$ is equal to $\max\{\alpha, \beta\}$ for all $1/2 \leq x \leq \frac{\max\{\alpha,\beta\}}{\alpha+\beta}$ and is equal to $(\alpha+\beta)x$ for any $x \geq \frac{\max\{\alpha,\beta\}}{\alpha+\beta}$. Thus, this function is monotone on the segment $[1/2; 1]$. Therefore, the function $\Theta(p_1, p_2, \ldots, p_m)$ is monotone in the variable p_m as a sum of functions of the type $f_{\alpha\beta}(x)$.

Moreover, the above equations show that

$$\Theta(p_1, p_2, \ldots, p_m) = 0.5 \sum_{e_1, \ldots, e_{m-1}} \max\{p_1^{(e_1)} \cdots p_{m-1}^{(e_{m-1})}, p_1^{(\bar{e}_1)} \cdots p_{m-1}^{(\bar{e}_{m-1})}\} = \Theta(p_1, \ldots, p_{m-1})$$

for any $p_m \le t$ where

$$t = \min_{e_1,\dots,e_{m-1}} \psi(p_1^{(e_1)} p_2^{(e_2)} \cdots p_{m-1}^{(e_{m-1})}, p_1^{(\bar e_1)} \cdots p_{m-1}^{(\bar e_{m-1})}). \ \square$$

6 Conclusion

We left open the following important question: how tight is the bound $O(\sqrt{\log n/n})$ in Theorem 1? That is, how fast can tend to zero the value $Q(\mathbf{p}, S_{mn}) - Q(\mathbf{p}, \tilde E_{\mathbf{p}})$ when n goes to infinity, where S be an arbitrary strategy (not necessary polynomial time).

Another question is: what is the performance of the following strategy $\tilde S$: given estimates $e_{11}, \dots, e_{mn}$ find those real number $p_1, p_2, \dots, p_m \in [1/2; 1]$ and binary values $v_1, v_2, \dots, v_n$ for which the probability of event "the experts with competence $p_1, p_2, \dots, p_m$ give estimates $e_{11}, \dots, e_{mn}$ provided the actual outcome sequence is v_1, v_2, …, v_n" is maximal. It is easy to see that this is again the maximal likelihood estimator. We do not know if the strategy $\tilde S$ is asymptotically optimal neither we know if it is polynomial time.

The third question is: what happens in the case when estimates of experts of the same outcome are correlated. What are reasonable assumptions about such correlation?

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7 Appendix. The proof of Lemmas 1 and 3

Let us prove Lemma 1. Recall that Lemma 1 states that $Q(\mathbf{p}, \tilde E_{\mathbf{p}}) \ge Q(\mathbf{p}, E)$ for all estimators E and all $\mathbf{p} = (p_1, p_2, \dots, p_m)$.

It is easy to see that

$$\tilde E_{\mathbf{p}}(\bar e_1, \dots, \bar e_m) = \overline{\tilde E_{\mathbf{p}}(e_1, \dots, e_m)}.$$

This implies that

$$\begin{aligned} &Q(\mathbf{p}, \tilde E_{\mathbf{p}}) \\ &= \mathrm{P}\big[\tilde E_{\mathbf{p}}(\theta_1^0, \dots, \theta_m^0) = 0\big] = \mathrm{P}\big[\tilde E_{\mathbf{p}}(\theta_1^1, \dots, \theta_m^1) = 1\big] \\ &= (1/2) \sum_{e_1,\dots,e_m} \mathrm{P}[\theta_1^{\tilde E_{\mathbf{p}}(e_1,\dots,e_m)} = e_1, \dots, \\ &\qquad \theta_m^{\tilde E_{\mathbf{p}}(e_1,\dots,e_m)} = e_m] \end{aligned} \quad (3)$$

Indeed, let $\oplus$ denote addition modulo 2. Then

$$\begin{aligned} &\mathrm{P}\big[\tilde E_{\mathbf{p}}(\theta_1^0, \dots, \theta_m^0) = 0\big] \\ &= \sum_{e_1,\dots,e_m : \tilde E(e_1,\dots,e_m)=0} p_1^{(e_1 \oplus 0)} \cdots p_m^{(e_m \oplus 0)} \\ &= \sum_{e_1,\dots,e_m : \tilde E(e_1,\dots,e_m)=1} p_1^{(\bar e_1 \oplus 0)} \cdots p_m^{(\bar e_m \oplus 0)} \\ &= \sum_{e_1,\dots,e_m : \tilde E(e_1,\dots,e_m)=1} p_1^{(e_1 \oplus 1)} \cdots p_m^{(e_m \oplus 1)} \\ &= \mathrm{P}\big[\tilde E_{\mathbf{p}}(\theta_1^1, \dots, \theta_m^1) = 1\big] \end{aligned} \quad (4)$$

The inequalities (4) imply that

$$Q(\mathbf{p}, \tilde E_{\mathbf{p}}) = (1/2) \sum_{e_1,\dots,e_m} p_1^{(e_1 \oplus \tilde E_{\mathbf{p}}(e_1,\dots,e_m))} \cdots p_m^{(e_m \oplus \tilde E_{\mathbf{p}}(e_1,\dots,e_m))}.$$

Let E be an estimator. Then

$$\begin{aligned} &Q(\mathbf{p}, E) \\ &\le 1/2\big(\mathrm{P}\big[E(\theta_1^0, \dots, \theta_m^0) = 0\big] + \mathrm{P}\big[E(\theta_1^1, \dots, \theta_m^1) = 1\big]\big) \\ &= 1/2 \sum_{e_1,\dots,e_m, v : E(e_1,\dots,e_m)=v} p_1^{(e_1 \oplus v)} \cdots p_m^{(e_m \oplus v)} \\ &= (1/2) \sum_{e_1,\dots,e_m} p_1^{(e_1 \oplus E(e_1,\dots,e_m))} \cdots p_m^{(e_m \oplus E(e_1,\dots,e_m))} \\ &\le (1/2) \sum_{e_1,\dots,e_m} p_1^{(e_1 \oplus \tilde E(e_1,\dots,e_m))} \cdots p_m^{(e_m \oplus \tilde E(e_1,\dots,e_m))} \\ &= Q(\mathbf{p}, \tilde E_{\mathbf{p}}) \end{aligned}$$

(the last equality is true by (3)). $\square$

Let us proof Lemma 3. Recall that Lemma 3 states that $\mathrm{P}[\tilde{E}_{\hat{\mathbf{p}}}(\theta_1, \dots, \theta_m) = 1] = Q(\mathbf{p}, \tilde{E}_{\hat{\mathbf{p}}})$ and that

$$| \, Q(\mathbf{p}, \tilde{E}_{\hat{\mathbf{p}}}) - Q(\mathbf{p}, \tilde{E}_{\mathbf{p}})| < m2^m\delta$$

provided $|\hat{p}_i - p_i| \leq \delta$ for all $i \leq m$.

The former property of the maximum likelihood estimator follows from equalities (4) if we substitute there $\tilde{E}_{\hat{\mathbf{p}}}$ for $\tilde{E}$.

Let the conditions of the second property are fulfilled. Then as in the proof of Lemma 1 we can prove that

$$\begin{aligned} &Q(\mathbf{p}, \tilde{E}_{\hat{\mathbf{p}}}) \\ &\quad = \; 1/2 \sum_{e_1,\dots,e_m} p_1^{(e_1 \oplus \tilde{E}_{\hat{\mathbf{p}}}(e_1,\dots,e_m))} \cdots p_m^{(e_m \oplus \tilde{E}_{\hat{\mathbf{p}}}(e_1,\dots,e_m))} \\ &Q(\mathbf{p}, \tilde{E}_{\mathbf{p}}) \\ &\quad = \; 1/2 \sum_{e_1,\dots,e_m} p_1^{(e_1 \oplus \tilde{E}_{\mathbf{p}}(e_1,\dots,e_m))} \cdots p_m^{(e_m \oplus \tilde{E}_{\mathbf{p}}(e_1,\dots,e_m))}. \end{aligned}$$

So it suffices to prove that for any $e_1, \dots, e_m$,

$$\begin{aligned} &\Big| p_1^{(e_1 \oplus \tilde{E}_{\hat{\mathbf{p}}}(e_1,\dots,e_m))} \cdots p_m^{(e_m \oplus \tilde{E}_{\hat{\mathbf{p}}}(e_1,\dots,e_m))} \\ &\quad - \; p_1^{(e_1 \oplus \tilde{E}_{\mathbf{p}}(e_1,\dots,e_m))} \cdots p_m^{(e_m \oplus \tilde{E}_{\mathbf{p}}(e_1,\dots,e_m))} \Big| \\ &\quad \leq 2m\delta. \end{aligned}$$

Let us fix arbitrary $e_1, \dots, e_m$. If

$$\tilde{E}_{\hat{\mathbf{p}}}(e_1, \dots, e_m) = \tilde{E}_{\mathbf{p}}(e_1, \dots, e_m)$$

we have nothing to do. Otherwise without loss of generality assume that

$$\tilde{E}_{\hat{\mathbf{p}}}(e_1, \dots, e_m) = 1, \quad \tilde{E}_{\mathbf{p}}(e_1, \dots, e_m) = 0.$$

This means that

$$p_1^{(e_1 \oplus 0)} \cdots p_m^{(e_m \oplus 0)} \geq p_1^{(e_1 \oplus 1)} \cdots p_m^{(e_m \oplus 1)}$$

and

$$\hat{p}_1^{(e_1 \oplus 0)} \cdots \hat{p}_m^{(e_m \oplus 0)} \leq \hat{p}_1^{(e_1 \oplus 1)} \cdots \hat{p}_m^{(e_m \oplus 1)}.$$

It is easy to see that

$$|\hat{p}_1^{(e_1 \oplus v)} \cdots \hat{p}_m^{(e_m \oplus v)} - p_1^{(e_1 \oplus v)} \cdots p_m^{(e_m \oplus v)}| \leq m\delta.$$

for any $v \in \{0, 1\}$. Therefore

$$|p_1^{(e_1 \oplus 0)} \cdots p_m^{(e_m \oplus 0)} - p_1^{(e_1 \oplus 1)} \cdots p_m^{(e_m \oplus 1)}| < 2m\delta. \; \square$$