On the structure of Ammann A2 tilings*

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Abstract

We establish a structure theorem for the family of Ammann A2 tilings of the plane. Using that theorem we show that every Ammann A2 tiling is self-similar in the sense of [B. Solomyak, Nonperiodicity implies unique composition for self-similar translationally finite tilings, Discrete and Computational Geometry 20 (1998) 265-279]. By the same techniques we show that Ammann A2 tilings are not robust in the sense of [B. Durand, A. Romashchenko, A. Shen. Fixed-point tile sets and their applications, Journal of Computer and System Sciences, 78:3 (2012) 731–764].

1 Introduction

There is a non-convex hexagon with right angles that has the following property. It can be cut into two similar hexagons so that the scaling factors are equal to ψ and ψ^2, where $\psi < 1$ (see Fig. 1). As the area of the original hexagon is equal to the sum of areas of the parts, the number ψ satisfies the equation

$$\psi^4 + \psi^2 = 1.$$

That is, ψ is the square root of the golden ratio: $\psi = \sqrt{\frac{\sqrt{5}-1}{2}}$.

The numbers on the sides in Fig. 1 indicate their lengths, which are powers of ψ. Using the equation $\psi^{n+2} + \psi^{n+4} = \psi^n$, it is easy to verify that the picture is consistent. Following Scherer [8], we will call any hexagon that is similar to that on Fig. 1 a *Golden Bee*. The *size* of a Golden Bee is defined as the length of its largest side.

We fix a positive real d and consider Golden Bees of sizes d and ψd as tiles (see Fig. 2), called *large* and *small* d-tiles. Tilings of the plane or its parts by these two tiles will be called *d-tilings*. If a d-tile P is cut into small and large $d\psi$-tiles, as shown in Fig. 1, then we

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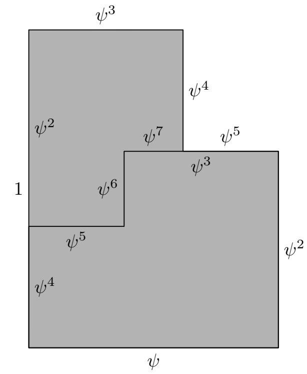


Figure 1: Cutting the Golden Bee into similar parts.

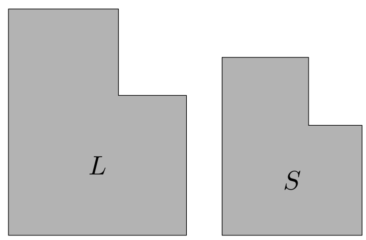


Figure 2: Ammann tiles.

call the large and small parts the *son* and the *daughter* of P, respectively. We also call the small part the *sister* of the large part and call the large part the *brother* of the small part.

For a d-tiling T we denote by σT the $d\psi$-tiling obtained from T by the substitution shown on Fig. 3: we cut each large d-tile in two smaller tiles, as shown on Fig. 1, and keep small d-tiles intact. Small d-tiles thus become large $d\psi$-tiles of the resulting $d\psi$-tiling. It is not hard to prove that σ is an *injective* mapping.

Grünbaum and Shephard [5] considered three families of d-tilings of the plane. Those families are defined by means of rules governing how one may attach tiles to each other when tiling the plane [5, Fig. 10.4.1(a)], [5, Fig. 10.4.1(c)] and [5, Fig. 10.4.1(d)]. All the three families are called "A2" and are attributed to Robert Ammann. The common name for these three families assumes that the families coincide. This is indeed true but is not evident and is not proven in [5] or elsewhere (we know that the families coincide from a personal communication of Korotin [6]). Yet another similar rule was introduced by Akiyama [1]. One can show [6] that the family of tilings satisfying Akiyama's rule coincides with the A2 family.

All the three rules of [5], as well as Akiyama's rule, imply the following *unique composition property*:

> *For each $d\psi$-tiling T' from A2 there is a (unique) d-tiling T in A2 such that $T' = \sigma T$.*

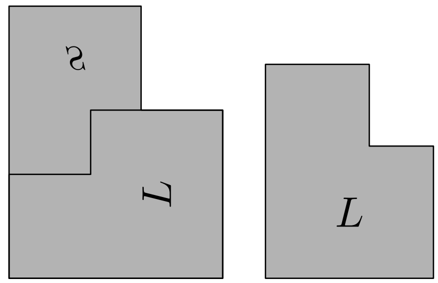


Figure 3: Substitution: every large tile (on the left) is cut into small and large tiles, and every small tile (on the right) becomes a large tile.

A well known "folklore" theorem (see [5, Theorem 10.1.1]) states that the unique composition property implies that all the tilings in the family are non-periodic[1]. Hence all A2 tilings are non-periodic.

In this paper, we focus on the first A2 family from [5, Chapter 10.4] defined by the following *Arrow rule* (see [5, Fig. 10.4.1(a)]):

> *Color in a given tiling the sides of large and small tiles, as shown in Fig. 4(b,c). The Arrow rule requires that for every pair of adjacent tiles each arrowed edge must fit against an edge with the same color pointing in the same direction.*

We will call tilings that satisfy this rule *A2 tilings*. Our main result describes the structure of A2 tilings of the plane (Theorem 3) in the following terms. A *supertile* is a tiling that is obtained from a single large tile by applying to it n substitutions for some natural n, which is called the *level* of the supertile (see Fig. 5). Each supertile tiles the tile from which it was produced by substitutions. An *infinite supertile* is a union of an infinite chain of supertiles

$$T_0 \subset T_1 \subset T_2 \subset \dots$$

such that for all n the tiling T_n tiles either the son, or the daughter of the tile tiled by T_{n+1} (see Fig. 6).

Our main Theorem 3 states that every A2 tiling of the plane is

- either an infinite supertile,
- or a union of 2 infinite supertiles S_1, S_2, which both tile half-planes obtained by cutting the plane by a line l; moreover, S_1, S_2 are reflections of each other in the axis l (see Fig. 7),

[1] *Sketch of proof.* Assume that a tiling T' from the family has a non-zero period t, that is $T' + t = T'$. Let T be the (unique) tiling from the family such that T' can be obtained from T by the substitution. Then t is a period of the tiling T as well. Indeed, since substitution and shift commute, the substitution applied to the tiling $T + t$ produces $T' + t$, which equals T' by assumption. The uniqueness implies that $T + t = T$. Similarly, T can by obtained by the substitution from another tiling from the family, which has also period t. In this way we can obtain tilings of the plane with tiles of arbitrarily large size whose period is t, which is obviously impossible.

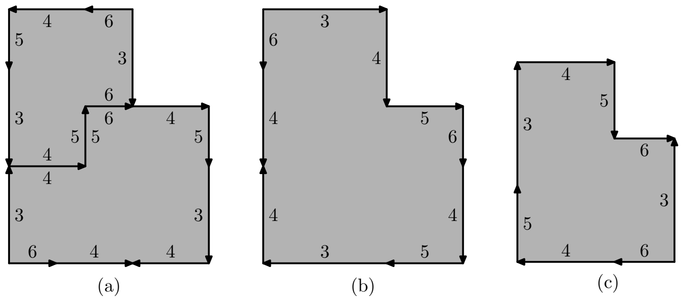


Figure 4: The Arrow rule for A2 tilings. The sides of tiles in this figure are divided into segments labeled by digits with arrows. Digits represent the colors and arrows identify orientations of segments. Digits correspond to the lengths of segments (i means the length proportional to ψ^i). Each arrowed edge must fit against an edge with the same label pointing in the same direction, e.g., as in (a).

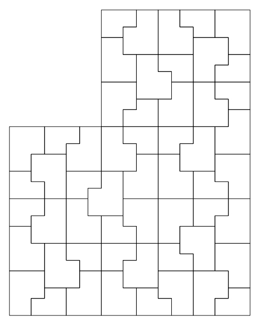


Figure 5: A supertile of level 8.

- or a union of 4 infinite supertiles S_3, S_4, S_5, S_6, which all tile quadrants obtained by cutting the plane by two orthogonal lines l_1, l_2; moreover, S_3 and S_4, as well as S_5 and S_6, are reflections of each other in the axis l_1 and S_3 and S_5 (S_4 and S_6) are reflections of each other in the axis l_2 (see Fig. 7).

Following [7], we then consider the family of self-similar tilings[2] associated with our substitution. A tiling T is called *self-similar* if any its finite pattern can be found in a supertile. It is not hard to show by induction that every supertile satisfies the Arrow rule and hence every self-similar tiling is an A2 tiling. Our second result states that the converse implication holds as well: every A2 tiling of the plane is self-similar (Theorem 4). This result follows from our first result on the structure of A2 tilings of the plane.

Finally, we answer the following question about "patching holes" in A2 tilings. Assume that a d-tiling T of the plane satisfies the Arrow rule everywhere except for a bounded region; is there an A2 d-tiling T' of the plane such that the symmetric difference of T and T' is finite? We show in Theorem 5 that this is not the case. This "patching holes" property

[2] *substitution tilings* in the terminology of [4]

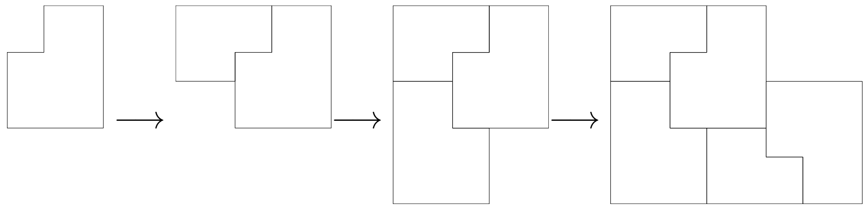


Figure 6: An infinite supertile is a union of a chain of supertiles.

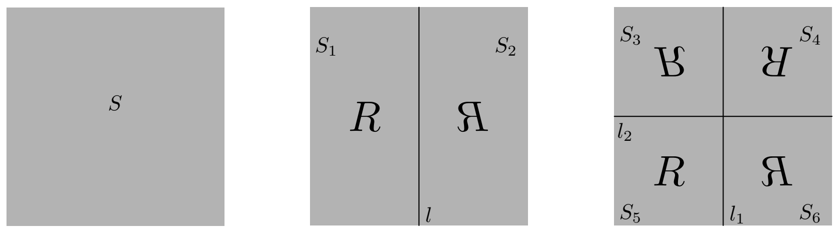


Figure 7: Three different types of A2 tilings of the plane. Here $S, S_1, S_2, S_3, S_4, S_5, S_6$ are infinite supertiles. The supertiles S_1 and S_2 are reflections of each other in the axis l. The supertiles S_3 and S_4 (S_5 and S_6) are reflections of each other in the axis l_1. The supertiles S_3 and S_5 (S_4 and S_6) are reflections of each other in the axis l_2.

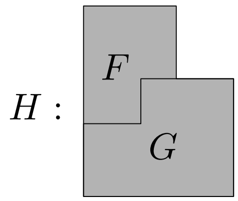
is interesting since it could make the tiling "robust" in the following sense: every tiling that has sufficiently sparse errors is close to a correct one. In [3] the patching holes property is used (together with self-similarity) to show that for some (artificial) tile set every tiling that is correct outside a "sparse enough" set is close to a tiling without errors. Therefore, if we would like to prove a similar result for a "geometric" tile set (e.g., Ammann, or Robinson, or Penrose tile sets) the first step could be to prove the patching hole property. Unfortunately, as Theorem 5 says, this is not possible for Ammann A2 tile set. We do not know any geometric tile set for which the patching hole property can be proven. On the other hand, we do not know any papers establishing negative results of this kind, e.g. for Robinson or Penrose tile sets.

The paper is organized as follows. In the next section we provide the main definitions. In Section 3 we state our results. In Section 4 we prove all theorems. The proofs of propositions and lemmas are deferred to Appendix.

2 Definitions

The notation $A \sqcup B$ refers to the disjoint union of A and B and $A \subset B$ means that A is a subset of B (not necessarily a proper subset).

Definition 1. A *tile of size* d is a Golden Bee of size d. We call tiles of size d *large d-tiles* and tiles of size ψd *small d-tiles*. A *tile* is a d-tile for some d. If a d-tile H is cut into small and large $d\psi$-tiles, F and G, as shown below,



then we call G and F the *son* and the *daughter* of H, respectively, we call F a *sister* of G and call G a *brother* of F. The form of Golden Bees ensures that each tile has a unique sister and a unique brother.[3]

Definition 2. *A d-tiling* is a non-empty set consisting of d-tiles that are pairwise disjoint (i.e., have no common interior points). A *tiling* is a d-tiling for some d. We denote by $[T]$ the set tiled by a tiling T.

Definition 3. The operation of *substitution* σ applied to a d-tiling T produces a $d\psi$-tiling that is obtained from T by cutting each large d-tile $A \in T$ into two tiles of sizes $d\psi$ and $d\psi^2$, as shown on Fig. 1 (page 2), and keeping all small d-tiles intact.[4] The resulting tiling is denoted by σT and is called the *decomposition* of T.

Since each tile has the unique brother, substitution is an injective operation. Indeed, if $\sigma T = T'$ for a d-tiling T, then T must consist of all d-tiles of the form ($F \cup$the brother of F), where F is a small $d\psi$-tile from T', and of all large $d\psi$-tiles $G \in T'$ whose sister is not in T'.

Definition 4. The inverse operation σ^{-1} is called *composition*.

This operation is not total, that is, some tilings have no compositions. For instance, if a $d\psi$-tiling T' consists of a single small $d\psi$-tile, then there is no d-tiling T with $\sigma T = T'$.

Definition 5. If $\sigma^{-1}(T)$ is defined, we say that T is *composable*. If $\sigma^{-n}(T)$ is defined for all natural numbers n, we say that T is *infinitely composable*.

Definition 6. A *d-supertile of level* $n \geqslant 0$ is the d-tiling $\sigma^n(\{H\})$ obtained by applying n substitutions to the initial d/ψ^n-tiling $\{H\}$ consisting of the single large d/ψ^n-tile H. (A supertile of level 8 is shown in Fig. 5 on page 4.) We will use also the notation $S_d(H)$ for $\sigma^n(\{H\})$ to indicate the size of tiles in $\sigma^n(\{H\})$. A *d-supertile of level* -1 is the d-tiling consisting of the single small d-tile.

It follows from the definition that every supertile of level $n \geqslant 0$ is composable and its composition is a supertile of level $n-1$. Every supertile of level $n \geqslant 1$ is a disjoint union of a supertile of level $n-1$ and a supertile of level $n-2$, see Fig. 8.

Definition 7. A d-tiling is called an *infinite d-supertile* if it is a union (= the limit) of an infinite chain of d-supertiles $T_0 \subset T_1 \subset T_2 \subset \dots$ such that for all n the tile $[T_n]$ is either the son, or the daughter of the tile $[T_{n+1}]$ (see Fig. 6 on page 5.)

[3]Indeed, if a small tile F and a large tile G are located as shown on the picture, then G can be identified by F, as the unique large tile whose angle formed by sides of length ψ^3 and ψ^6 fills the cavity of F in such a way that the side of length ψ^5 is shared by the side of small tile of the same length. In a similar way the tile F can be identified by G.

[4]It is more common in the literature to inflate the initial tiling by $1/\psi$ before substitution so that the resulting tiling is again a d-tiling.

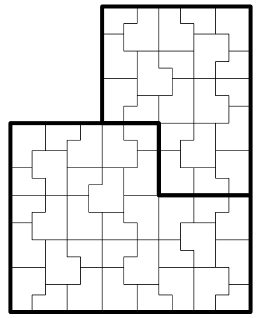


Figure 8: The supertile of level 8 is represented as a union of supertiles of levels 7 and 6.

It is not hard to see that every infinite supertile is infinitely composable.

Definition 8. Tilings T and S are called *congruent* if there is an isometry f of the plane such that $T = \{f(H) \mid H \in S\}$.

3 Results

Our goal is two-fold: we want to understand how A2 tilings of the plane may look like and, using that understanding, to prove some their properties. It turns out that our technique works for tilings of any convex set, therefore we state our theorem for tiling of arbitrary convex sets (actually, we will see that, among convex sets, A2 tilings can tile only a plane, a half-plane or a quadrant).

The next proposition establishes some relations between the notions of a supertile, an A2 tiling and an infinitely composable tiling.

Proposition 1 ([5, 2])**.** *(a) Every (finite or infinite) supertile is an A2 tiling. (b) Each A2 tiling of a convex set is composable. (c) The composition of every A2 tiling of a convex set is again an A2 tiling (hence every A2 tiling of a convex set is infinitely composable).*

For the sake of completeness we present a proof of this proposition in Appendix.

3.1 The structure of A2 tilings of convex sets

The structure of A2 tilings of convex sets is established in Theorems 2 and 3 below. The first theorem applies to all infinitely composable tilings. The second one applies only to A2 tilings. Both theorems express possible structures of tilings in terms of infinite supertiles. Thus it is useful to understand how infinite supertiles may look like. Therefore we start with a description of infinite supertiles.

Recall that an infinite d-supertile is a union of a chain of d-supertiles

$$T_0 \subset T_1 \subset T_2 \subset \dots.$$

such that $[T_n]$ is either the son, or the daughter of $[T_{n+1}]$ for all n. W.l.o.g. we may assume that the supertile T_0 consists of a single tile. In this case we will call the sequence of tiles $[T_0], [T_1], [T_2], \dots$ a *representation* of the infinite d-supertile $\bigcup_{n=0}^{\infty} T_n$. This definition can be applied to finite supertiles as well, in which case the sequence is finite. It is not hard to see that for every sequence of tiles $H_0, H_1, H_2, \dots$ such that H_n is the son or the daughter of H_{n+1} (for all n) there is a unique infinite supertile with representation $H_0, H_1, H_2, \dots$.

A supertile can have many representations. More specifically the following proposition holds.

Proposition 2. *(a) Assume that T is an infinite d-supertile and H is any of its tiles. Then there is a unique representation $H_0, H_1, H_2, \dots$ of T with $H_0 = H$. (b) For any two representations $H_0, H_1, H_2, \dots$ and $G_0, G_1, G_2, \dots$ of an infinite supertile T there are n, m such that $H_{i+n} = G_{i+m}$ for all $i \geqslant 0$ (the representations have common tail). (c) If a tiling T is infinitely composable and H is any of its tiles, then there is a unique infinite supertile S with $H \in S \subset T$.*

Corollary 1. *Any infinite supertile T has only trivial symmetry (if f is an isometry such that $f(T) = T$, then f is the identity mapping).*

Proof. Let H be any tile from T and let $H_0, H_1, H_2, \dots$ be the unique representation of T with $H_0 = H$. Then $f(H_0), f(H_1), f(H_2), \dots$ is a representation of $f(T) = T$. By Proposition 2(b) we have $H_n = f(H_m)$ for some m, n. As f does not change the size of tiles, we must have $m = n$ and hence $H_n = f(H_n)$. Since the Golden Bee has only trivial symmetry, f is the identity mapping. □

Every representation $H_0, H_1, H_2, \dots$ of an infinite d-supertile T is completely specified by the initial tile H_0 and the infinite sequence α of letters s, l where $\alpha_n = s$ if H_n is the daughter of H_{n+1} and $\alpha_n = l$ if H_n is the son of H_{n+1} (s, l stand for "small" and "large"). The pair (H_0, α) will be called a *succinct representation* of the infinite d-supertile T. Now we formulate a simple criterion of whether two infinite supertiles are congruent.

Definition 9. Define *the weighted length* of a sequence u of letters s, l by the formula:

$$w(u) = 2(\text{the number of } s\text{'s in } u) + (\text{the number of } l\text{'s in } u).$$

Infinite sequences α and β of letters s, l are *equivalent* iff α and β can be represented as concatenations $\alpha = u\gamma$ and $\beta = v\gamma$ for some finite sequences u, v with $w(u) = w(v)$.

The weighted length has the following meaning: if (H, u) is a succinct representation of a finite d-supertile S, then the level of S is equal to $w(u)$, if H is a large d-tile, and to $w(u) - 1$ otherwise (see Fig. 9).

Proposition 3. *Assume that H and G are large d-tiles. Infinite d-supertiles with succinct representations (H, α) and (G, β) are congruent iff α and β are equivalent.*

This proposition can be easily generalized to the case when H and G are small tiles, or a large tile and a small tile.

The next theorem explains which part of the plane tiles an infinite supertile with succinct representation (H, α):

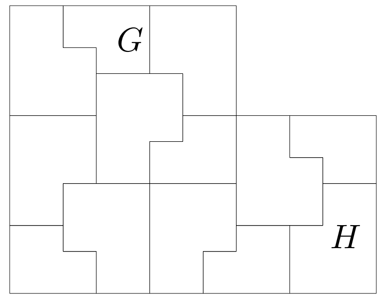


Figure 9: The supertile of level 5 has succinct representations (H, sls) and $(G, ssll)$, where $w(sls) = 2+1+2 = 5$ and $w(ssll) = 2+2+1+1 = 6$.

Theorem 1. *(a) An infinite supertile with succinct representation (H, α) does not tile the entire plane iff a tail of α consists of the blocks s and lsl. (b) In this case (when S does not tile the plane) it tiles a half-plane or a quadrant; more specifically, it tiles a quadrant iff a tail of α consists of alternating blocks s–lsl–s–lsl–....*

Now we present a description of infinitely composable tilings.

Theorem 2. *Every infinitely composable tiling can be represented as a disjoint union of up to four infinite supertiles; such a representation is unique.*

By Proposition 1 every A2 tiling of a convex set is infinitely composable. Therefore this theorem applies also to arbitrary A2 tiling of convex sets. Theorem 2 and Theorem 1 imply that such a set may be either the entire plane, or a half-plane, or a quadrant. Indeed, these are the only convex sets which are disjoint unions of quadrants, half-planes and planes. Moreover, A2 tiling of convex sets have the following important property: if such a tiling consists of more than one supertile then those supertiles must be axial symmetrical.

Theorem 3. *If an A2 tiling tiles a convex set, then that set is either a plane, or a half-plane, or a quadrant.*
(a) Every A2 tiling of the entire plane is either a supertile, or a disjoint union of A2 tilings of half-planes; in the second case those tilings of half-planes are axial symmetrical in the line that separates the half-planes.
(b) Every A2 tiling of a half-plane is either an infinite supertile, or a disjoint union of A2 tilings of quadrants; in the second case those tilings of quadrants are axial symmetrical in the line that separates the quadrants.
(c) For every d there are three different d-tilings of a quadrant (up to congruence) and they all are infinite supertiles.

Remark 1. By this theorem every A2 tiling T of the plane has one and only one of the forms shown on Fig. 7 on page 5. That form can be determined by any succinct representation of any infinite supertile $S \subset T$. Thus there is a natural 1-1 correspondence between A2 tilings of the plane (we identify here congruent tilings) and equivalence classes of infinite l-s-sequences.

3.2 A2 tilings = self-similar tilings

Now we proceed to our second result, which shows that every A2 tiling of a convex set is self-similar.

Definition 10. A *pattern* is a finite tiling. A pattern is *legal* if it is a subset of a supertile. A tiling T is called *self-similar* (with respect to the substitution shown on Fig. 3 on page 3) if all its finite subsets are legal.

By Proposition 1 every supertile is an A2 tiling. Hence every self-similar tiling is an A2 tiling. Thus we have the following inclusions for tilings of convex sets: T is a self-similar tiling $\Rightarrow$ T is an A2 tiling $\Rightarrow$ T is infinitely composable. The second implication is not invertible (see Example 1 on page 16). Our second result states that the first implication is.

Theorem 4. *Every A2 tiling of a convex set is self-similar.*

This theorem is not straightforward, as one might think. We derive this theorem from Theorem 3. One could try to prove it directly. We outline a sketch of such proof and point out the problems we face.

Sketch of proof. Let T be a given A2 tiling of a convex set. We want to show that it is self-similar. By Proposition 1 the tiling T is infinitely composable. We first represent T as a disjoint union of infinite supertiles. To this end we apply compositions to T and consider for each n the tiling $\sigma^{-n}T$.

Pick any tile H from T. For every n, the tile H is covered by a tile H_n from $\sigma^{-n}T$. Consider the tiling S_n that consists of all tiles from T that are covered by H_n. Then S_n is a supertile. The supertiles S_n form a chain $S_0 = \{H\} \subset S_1 \subset \dots \subset S_n \subset \dots$ and their union is an infinite supertile. As all supertiles S_n are self-similar, so is their union $S = \bigcup_{n=0}^{\infty} S_n$. If it happens that $S = T$, then we are done.

Otherwise, if $S \neq T$, starting from any tile B from the difference $T \setminus S$, we can find a new infinite supertile $S' \subset T$ which contains B. It is not hard to see that S' and S are disjoint (indeed, if they shared a tile C, then both supertiles S, S' could be constructed starting from C as well and hence S and S' would coincide). In this way we can represent the given tiling in the form $T = S_1 \sqcup S_2 \sqcup \dots$, where $S_1, S_2, \dots$ are infinite supertiles.

Now we face the following problem. We have to show that every pattern $W \subset T$ is legal. It may happen that W intersects different S_i's from the representation $T = S_1 \sqcup S_2 \sqcup \dots$. In this case self-similarity of S_i's does not imply that W is legal. We solve this problem as follows.

Up to now we have only used infinite composability of T. As we have said, there is an infinitely composable tiling of the plane which is not self-similar (Example 1 on page 16). So we have to use our assumption that T satisfies the Arrow rule. Here Theorem 3 comes into play. By that theorem the tilings S_i are mirror images of each other as on Fig. 7 (page 5). Moreover, it follows from the proof of Theorem 3 that in the case $T = S_1 \sqcup S_2$ the pattern W is covered by two symmetrical supertiles A_1, A_2 and in the case $T = S_1 \sqcup S_2 \sqcup S_3 \sqcup S_4$ the pattern W is covered by four symmetrical supertiles B_1, B_2, B_3, B_4, as shown on Fig. 10. It remains to note that both patterns "two large tiles sharing their backs so that they are

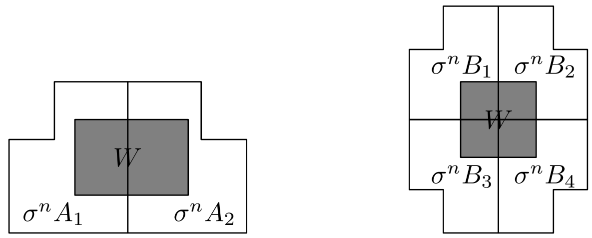


Figure 10: On the left: the pattern W is covered by two symmetrical supertiles A_1, A_2. On the right: the pattern W is covered by four symmetrical supertiles B_1, B_2, B_3, B_4.

reflections of each other (as in Fig. 10 on the left)" and "four large tiles sharing their backs and bottoms so that they are reflections of each other (as in Fig. 10 on the right)" are legal. Indeed, they appear in the supertile of level 8 shown on Fig. 5 (page 4). Hence applying to the supertile of level 8 the appropriate number of substitutions we get a supertile that includes W. □

Remark 2. As the family of self-similar tilings coincide with the family of A2 tilings, Theorem 3 applies to self-similar tilings as well. However, the direct proof of Theorem 3 for self-similar tilings is only a little bit easier than ours (namely, the proof of Proposition 1 is a bit simpler for self-similar tilings).

3.3 Non-robustness of A2 tilings

Our third result states that A2 tilings are sensitive to errors in the following sense. Durand, Romashchenko and Shen [3], who considered tilings of the plane by square tiles, defined the following notion of robust families of tilings. Assume that we have a set τ of tiles, where each tile is a square of size 1×1 with colored edges. Consider the family $\mathcal{T}$ consisting of tilings of parts of plane in which tiles can be attached only side-by-side so that the colors match. The family $\mathcal{T}$ is called *robust* if for any large enough tiling of a set with a hole one can "patch the hole", that is, one can find a tiling of the same set plus the hole which differs from the original tiling not much.

Definition 11. Let $c_1 < c_2$ be positive integers. We say that a family of tilings $\mathcal{T}$ is (c_1, c_2)-*robust* if the following holds: For every positive natural Δ and for every tiling $T \in \mathcal{T}$ that tiles a set

$$S \supset ([-c_2\Delta, c_2\Delta] \times [-c_2\Delta, c_2\Delta]) \setminus ([-\Delta, \Delta] \times [-\Delta, \Delta])$$

there exists a tiling $T' \in \mathcal{T}$ of the set $S \cup ([-\Delta, \Delta] \times [-\Delta, \Delta])$ that contains all tiles from T lying outside of the square $[-c_1\Delta, c_1\Delta] \times [-c_1\Delta, c_1\Delta]$.

The smaller c_1, c_2 are the stronger this definition is. Durand, Romashchenko and Shen [3] exhibited a family of non-periodic tilings with many interesting properties that is (c_1, c_2)-robust for some c_1, c_2.

The notion of a robust family naturally generalizes to tilings by arbitrary tiles (of any shape) defined by arbitrary local rules (like, say, the Arrow rule). In this paper, we show that the family of A2 tilings is *not* (c_1, c_2)-robust for all c_1, c_2. Moreover, the following is true:

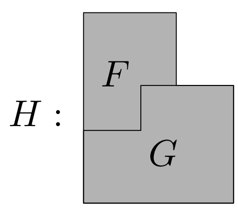
Theorem 5. *There is a tiling T of the plane that satisfies the Arrow rule everywhere except a bounded region and that has the following property: for any A2 tiling T' of the plane the difference $T \setminus T'$ is infinite.*

Corollary 2. *The family of A2 tilings is* not *(c_1, c_2)-robust for all c_1, c_2.*

Proof of the corollary. Let T be the tiling from the theorem and c_1, c_2 arbitrary natural numbers. Remove from T all tiles violating the Arrow rule. We obtain an A2 tiling of a set S, which is equal to the plane minus a bounded hole H. Let Δ be equal to the diameter of the hole H and hence S includes $[-c_2\Delta, c_2\Delta]^2 \setminus [-\Delta, \Delta]^2$. Assume now that an A2 tiling T' tiles the set $S \cup [-\Delta, \Delta]^2$, that is, the entire plane. We have to show that T' does not contain a tile from T lying outside of the square $[-c_1\Delta, c_1\Delta]^2$. By Theorem 5 the difference $T \setminus T'$ is infinite and hence at least one its tile lies outside that square. $\square$

4 Proofs of theorems

In this section we prove all theorems. The proofs of propositions and lemmas are deferred to Appendix. Several times in the proofs, we will apply composition to a part S of a tiling T and conclude that $\sigma^{-1}T$ includes $\sigma^{-1}S$. In general we cannot make such conclusion, as the following example demonstrates. Let $S = \{G\}$ and $T = \{F, G\}$, where F, G are the daughter and the son of a large d/ψ-tile H:



Then $\sigma^{-1}S = \{G\}$ and $\sigma^{-1}T = \{H\}$. Thus $S \subset T$ while $\sigma^{-1}S \not\subset \sigma^{-1}T$. However, this may happens only when S contains a large tile G whose cavity is not covered by $[S]$. This makes possible for T to include the sister of G, in which case G produces different tiles in $\sigma^{-1}S$ and in $\sigma^{-1}T$.

A composable tiling S is called *proper* if the cavity of every large tile from S is covered by $[S]$. The following tilings are proper:

- Every supertile of level $n > 0$ is proper.
- More generally, every tiling of the form σ^2T is proper, where T is any tiling. (Indeed, every large tile from σ^2T is either a small tile from σ^1T, in which case its cavity is covered by its brother from σ^1T, or it is the brother of a small tile from σ^2T, which covers its cavity.)

- Every tiling of a convex set is proper.

For proper tiling we have the following

Lemma 1. *(a) If a proper tiling S is a subset of a composable tiling T, then $\sigma^{-1}S \subset \sigma^{-1}T$. (b) If T, S are proper tilings then $\sigma^{-1}(S \cup T) = \sigma^{-1}S \cup \sigma^{-1}T$.*

For reader's convenience, the following diagram represents the dependencies in the proofs:
Lemma 1 $\Rightarrow$ Proposition 2 $\Rightarrow$ Proposition 3.
Proposition 2 and Theorem 1 $\Rightarrow$ Theorem 2.
Proposition 1, Proposition 3, Theorem 1, and Theorem 2 $\Rightarrow$ Theorem 3.
Proposition 1 and Theorem 3 $\Rightarrow$ Theorem 4.
Lemma 1 $\Rightarrow$ Theorem 5.

4.1 Proof of Theorem 1

Let S be an infinite supertile with succinct representation (H_0, α). Consider two transformations s, l of tiles: $s(H)$ is the unique tile whose daughter is H, and $l(H)$ is the unique tile whose son is H. Define $H_{i+1} = s(H_i)$ if the ith letter of α is s and $H_{i+1} = l(H_i)$ otherwise. Then $H_0, H_1, H_2, \dots$ is a representation of S.

(a) Assume that S does not tile the entire plane. The sides of hexagons H_n stretch in two directions. Call those directions *horizontal and vertical* directions.

Claim. *If the part of the plane tiled by S intersects a vertical line and intersects a horizontal line, then it includes the common points of the lines.*

Proof. For some i both lines intersect H_i. This does not imply yet that H_i contains their common point, as it might happen that it falls into the cavity of H_i. However in this case it falls into H_{i+1}. □

By this claim every proper subset of the plane tiled by an infinite supertile does not intersect a vertical or a horizontal line, call that line L. Then S lies in one of the two half-planes defined by the line L.

Consider the distance δ_i from the tile H_i to the line L. As H_{i+1} covers H_i, the sequence $\{\delta_i\}$ is non-increasing. Moreover, as i is incremented by 1, the distance either remains the same, or decreases by some positive constant ε (or more). Hence starting from some i the distance does not change: there are δ and k such that $\delta_i = \delta$ for all $i \geqslant k$.

Shift the line L towards the set $[S] = \bigcup_{n=0}^{\infty} H_n$ at the distance δ. Now L touches all tiles H_i with $i \geqslant k$. W.l.o.g. we may assume that L is a horizontal line and that all H_i lie above L. For all $i \geqslant k$ the tile H_i has a point on the line L and hence an entire side of the tile H_i lies on the line L.

We will view Golden Bees as "chairs" having the top, the back, the front, and the bottom (see Fig. 11). The side of H_i lying on the line L can be either the front side, or the back side, or the bottom side, or the top side (see Fig. 12). We need first to understand, how that side changes as i increments. The following diagram shows how transformations s and l change the sides of tiles lying on the line L:

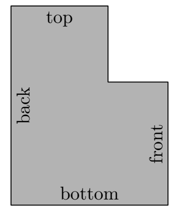


Figure 11: Then names of sides of a tile

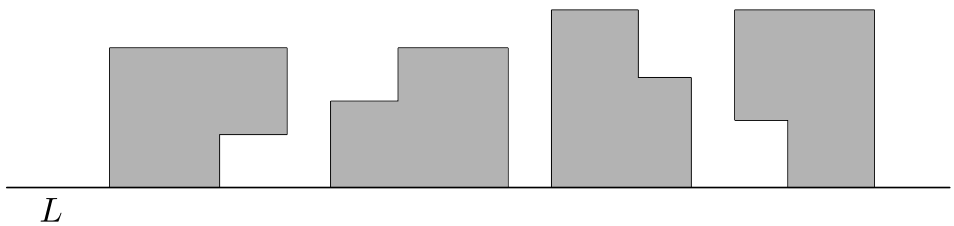
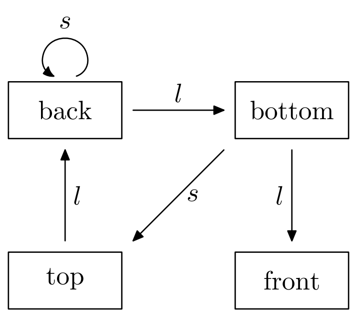


Figure 12: Tiles lie on the horizontal line L on the front side, on the back side, on the bottom side and on the top side.



For example, the transition (bottom $\xrightarrow{l}$ front) means that, if the line L contains the bottom side of a tile H, then the front side of $l(H)$ lies on L. This fact is easy to verify by observing the cut in Fig. 1 (page 2). If a transition is absent in this table, then such case is impossible. For instance, if H lies on the line L on its top side, then the tile $s(H)$ crosses the line L.

This diagram can be viewed as a finite automaton. That automaton has the following property: in whatever state we start and whatever infinite sequence of transitions we follow, we will always pass through the "back" state. Thus, for some $i \geqslant k$ the back side of H_i must lie on the line L. Moreover, there are infinitely many such i's and between any two consecutive such i's only the transition s or the sequence of transitions lsl may occur. This completes the 'only if' part of Theorem 1(a).

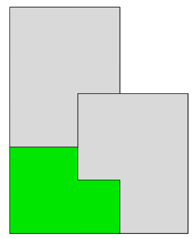
Remark 3. It follows from the above argument, that if an infinite supertile S with representation $H_0, H_1, H_2, \ldots$ does not tile the entire plane, then for infinitely many n the back side of the tile H_n lies on the border line of the area tiled by S.

Conversely, assume that a tail of α consists of blocks s and lsl. We have to show that the infinite supertile S with succinct representation (H_0, α) does not tile the entire plane.

W.l.o.g. we may assume that α itself consists of blocks s and lsl. Consider the line passing through the back side of the tile H_0. Then all tiles H_n lie in the same half-plane as H_0 does. Hence the tiling S tiles at most a half-plane.

(b) Let S be an infinite supertile with succinct representation (H_0, α). Assume that $\alpha = u\beta$ where β consists of the alternating blocks s and lsl. In other words, β consists of the alternating s and l.

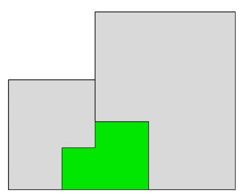
Let us show that S tiles a quadrant. Assume first that u is empty. The mapping sl transforms the small green (gray in the black and white image) tile into a large tile that is inscribed in the same quadrant.



Therefore infinite number of applications of the transformation sl fills up the quadrant but not more. If u is not empty, then the same arguments apply to some tile H_n from the representation of S.

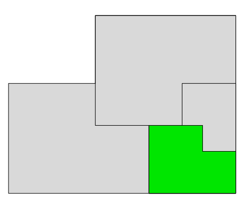
Assume now that a tail of α consists of the blocks s and lsl but they *do not* alternate. That is, the tail has infinitely many occurrences of ss or infinitely many occurrences of $lsllsl$.

Transformation s maps a tile that lies on a line on its back to a larger tile that also lies on the same line on its back. The second application of s increases the part of the tile that belongs to the line in the other direction.



Thus if a tail of α has infinitely many occurrences of ss (and consists of blocks s and lsl) then its application to the initial tile fills up a half-plane.

Similar arguments apply when α has infinitely many of occurrences of $lsllsl$. The mapping lsl also maps a tile that is attached to a line by its back to a larger tile that is again attached to the same line by its back.



Thus the double application of lsl increase the area of attachment in both directions. Theorem 1 is proved.

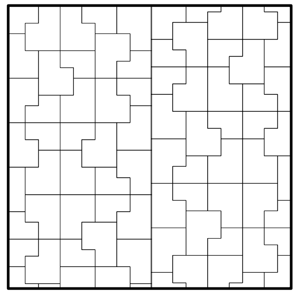


Figure 13: An infinitely composable tiling of the plane which is not A2.

Remark 4. It follows from the above arguments, that if an infinite supertile with representation $H_0, H_1, H_2, \dots$ tiles a quadrant, then for all large enough n the back and bottom of the tile H_n lie on the boundary of that quadrant.

Example 1. Let S be an infinite supertile tiling the half-plane and let S' be its reflection in the border line l of the half-plane. Shift S' a little bit along l (see Fig. 13). The resulting tiling is an infinitely composable tiling of the plane and is not an A2 tiling.

4.2 Proof of Theorem 2

Let T be a given infinitely composable tiling. We show first that T can be represented in a unique way as a disjoint union of infinite supertiles.

Let H be a large tile from T. By Proposition 2(c) there is a unique infinite supertile $S_1 \subset T$ containing H. If S_1 coincides with T then T is an infinite supertile. Otherwise pick any tile G in $T \setminus S_1$. Again by Proposition 2(c) there is a unique infinite supertile $S_2 \subset T$ containing G. The supertiles S_1, S_2 are disjoint. Indeed, assume that they share a tile K. Then we have both $K \in S_1 \subset T$ and $K \in S_2 \subset T$, which contradicts the uniqueness part of Proposition 2(c)

As each infinite supertile covers at least a quadrant, in this way we can represent T as a disjoint union of up to four infinite supertiles. Such representation is unique, as we have already shown that any two intersecting infinite supertiles that are subsets of T coincide.

4.3 Proof of Theorem 3

Let S be an A2 tiling of a convex set. By Proposition 1 it is infinitely composable. Thus by Theorem 2 the tiling S can be represented in a unique way as a disjoint union up to four infinite supertiles, and each of them tiles either the entire plane, or a half-plane, or a quadrant. If a convex set is a disjoint unions of quadrants and half-planes, then it is either a plane, or a half-plane, or a quadrant. This proves the first statement in Theorem 3.

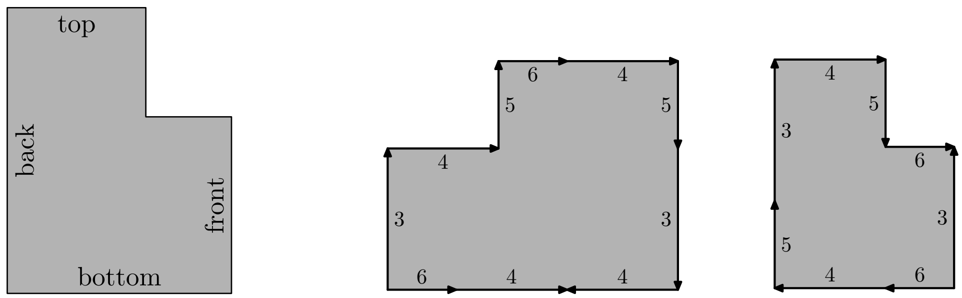
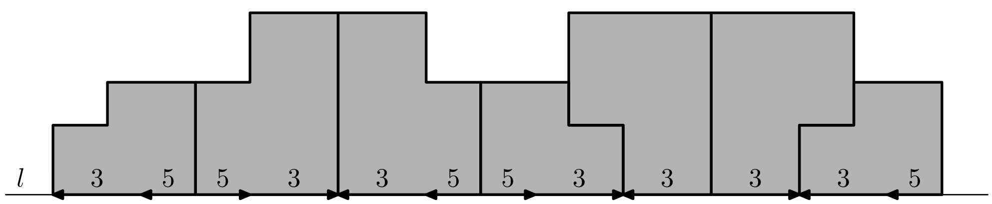


Figure 14: Coloring of large and small tiles.

4.3.1 Proof of Theorem 3(a)

Let S be an A2 tiling of the plane. As we have just seen, S is either a supertile, or a disjoint union of A2 tilings T, R of half-planes. We have to show that in the second case tilings T, R are axial symmetrical in the line l that separates the half-planes. Let R' be the reflection of R in the axis l. Then both T, R' tile the same half-plane and we have to show that they coincide.

Call the set of colored oriented segments of tiles in T lying on l the *shadow* of T



For instance, the sequence

$$\overleftarrow{3}\,\overleftarrow{5}\,\overrightarrow{5}\,\overrightarrow{3}\,\overleftarrow{3}\,\overleftarrow{5}\,\overrightarrow{5}\,\overrightarrow{3}\,\overleftarrow{3}\,\overrightarrow{3}\,\overleftarrow{3}\,\overleftarrow{5}$$

is the shadow of the tiling from the above picture. Each segment in the shadow is identified by the triple (start point, end point, color). It suffices to prove the following

Lemma 2. *Given the shadow of an A2 tiling T of a half-plane we can reconstruct the tiling.*

(Indeed, the shadows of T and R' coincide, as $T \cup R$ is an A2 tiling. Thus by the lemma we have $T = R'$ and hence R is the reflection of T.)

Proof of Lemma 2. Obviously only front, back, bottom and top sides of tiles can lie on border line of the half-plane tiled by T. A quick look at the coloring of large and small tiles reveals that these sides of tiles consists of blocks $\overrightarrow{6}\,\overrightarrow{4}$, $\overrightarrow{5}\,\overrightarrow{3}$, $\overrightarrow{3}$, $\overrightarrow{4}$ (see Fig. 14). Thus all shadows consist of these blocks. Actually, blocks with odd numbers cannot occur together with blocks with even numbers.

Claim 1. *Every shadow either consists either of blocks $\overrightarrow{6}\,\overrightarrow{4}$, $\overrightarrow{4}$, or of blocks $\overrightarrow{5}\,\overrightarrow{3}$, $\overrightarrow{3}$.*

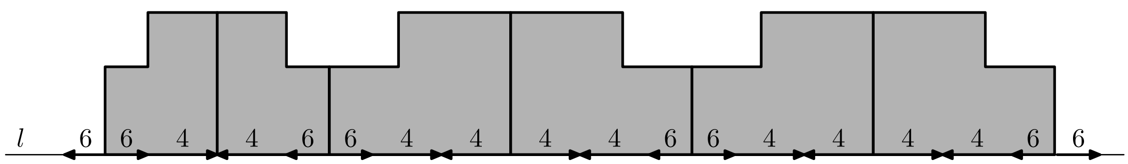
Proof. Every tile has three sides that have colors $\overrightarrow{3}$ and $\overrightarrow{5}$ and three sides that have colors $\overrightarrow{4}$ and $\overrightarrow{6}$. Call the sides of the first type *odd* and the sides of the second type *even*. Every even side of a hexagon is parallel to every its even side and is orthogonal to every its odd side.

In every A2 tiling every two adjacent hexagons G, H share a colored segment. Thus they *have the same orientation*: odd sides of G are parallel to odd sides of H and are orthogonal to even sides of H. This implies that all hexagons in an A2 tiling of a convex set have the same orientation. As all sides of the given tiling T that lie on its border l are parallel to each other, either they all are even sides, or they all are odd sides. □

Let us show first how to reconstruct from the shadow all the tiles from T that are adjacent to the border line l of the half-plane. Assume first that the shadow of T consists of blocks $\overrightarrow{6}\overrightarrow{4}$, $\overrightarrow{4}$.

Claim 2. *Given a shadow of T consisting of blocks $\overrightarrow{6}\overrightarrow{4}$, $\overrightarrow{4}$, we can reconstruct all the tiles from T that are adjacent to the border line l of the half-plane.*

Proof. The given shadow can consist of fronts and backs of large tiles and tops and bottoms of small tiles (see Fig. 14). However, if a large tile H lies on a line on its front side, then both tiles $s(H)$, $l(H)$ cross that line (recall the diagram on page 14). Similarly, if a small tile H lies on a line on its top side, then the tile $s(H)$ crosses that line. Hence the given shadow consists of bottoms of small tiles and backs of large tiles.



At the end of the bottom $\overrightarrow{6}\overrightarrow{4}$ of each small tile there is an orthogonal side (of the same tile) $\overrightarrow{5}\overrightarrow{3}\uparrow$ pointing *to* the interior of the half-plane (this is easily verified by examining Fig. 14). Only the back of another small tile can match that block $\overrightarrow{5}\overrightarrow{3}\uparrow$ and thus the bottom of each small tile $\overrightarrow{6}\overrightarrow{4}$ must be followed by the symmetrical block $\overleftarrow{4}\overleftarrow{6}$. On the other hand, at the end of the back side $\overrightarrow{6}\overrightarrow{4}\overleftarrow{4}$ of every large tile there is a side $\overleftarrow{3}\overleftarrow{5}\downarrow$ (of the same tile) pointing *from* the interior of the half-plane (this is easily verified by examining Fig. 14). Only the bottom of another large tile can match that block $\overleftarrow{3}\overleftarrow{5}\downarrow$ and thus the back $\overrightarrow{6}\overrightarrow{4}\overleftarrow{4}$ of every large tile must be followed by the block $\overrightarrow{4}\overleftarrow{4}\overleftarrow{6}$.

This analysis shows that the shadow can be divided into blocks $\overrightarrow{6}\overrightarrow{4}\overleftarrow{4}\overleftarrow{6}$ and $\overrightarrow{6}\overrightarrow{4}\overleftarrow{4}\overrightarrow{4}\overleftarrow{4}\overleftarrow{6}$. Such division is unique, as the arrow on every digit 6 shows the direction to the block starting or ending by that digit. We must attach to every block $\overrightarrow{6}\overrightarrow{4}\overleftarrow{4}\overleftarrow{6}$ a pair of small tiles lying on l on their tops and sharing their backs and to every block $\overrightarrow{6}\overrightarrow{4}\overleftarrow{4}\overrightarrow{4}\overleftarrow{4}\overleftarrow{6}$ a pair of large tiles lying on l on their backs and sharing their bottoms. □

A similar lemma (with a similar proof) holds also for 3-5-shadows. However we do not need it, as we can finish the proof as follows.

Using the procedure of Claim 2, we are able to reconstruct a given tiling T from its shadow in an arbitrarily large stripe along the border. Indeed, to reconstruct T in the stripe of width $d\psi^{2-i}$ near the border line, first find the shadow of the tiling $\sigma^{-i}T$ obtained from T by i compositions. Examining Fig. 4 (page 4), it is not hard to verify that the substitution transforms oriented colored segments according to the following rules

$$\begin{aligned} \overrightarrow{6} &\to \overrightarrow{5} \\ \overrightarrow{5} &\to \overrightarrow{4} \\ \overrightarrow{4} &\to \overrightarrow{3} \\ \overrightarrow{3} &\to \overleftarrow{4}\,\overleftarrow{6}\,. \end{aligned}$$

Applying to the given shadow the inverse map i times (every $\overleftarrow{4}$ followed by $\overleftarrow{6}$ is replaced using the last line and remaining $\overleftarrow{4}$'s are replaced using the second line), we are able to find the shadow of the tiling $\sigma^{-i}T$. If it happens to be a 3-5-shadow then apply composition one more time. Then by the procedure of Claim 2 we reconstruct the tiling $\sigma^{-i}T$ (or $\sigma^{-i-1}T$) in the stripe of width $d\psi^{2-i}$ (or $d\psi^{2-i-1}$) near the border of the half-plane. Finally, apply i (or $i+1$) substitutions to the obtained tiling.

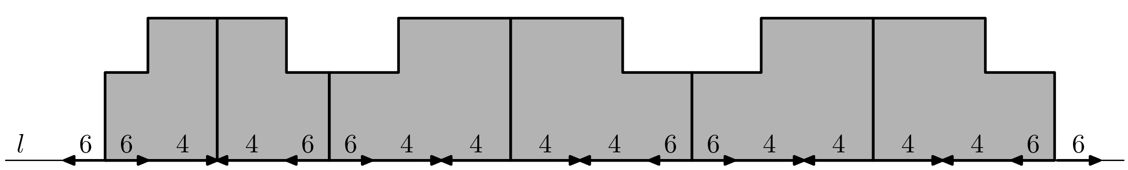
For instance, assume that we are given the shadow

$$\dots\ \overleftarrow{4}\,\overrightarrow{4}\,\overleftarrow{4}\,\overleftarrow{6}\,\overrightarrow{6}\,\overrightarrow{4}\,\overleftarrow{4}\,\overrightarrow{4}\,\overleftarrow{4}\,\overleftarrow{6}\,\overrightarrow{6}\,\overrightarrow{4}\,\overleftarrow{4}\,\overleftarrow{6}\,\overrightarrow{6}\,\overrightarrow{4}\,\overleftarrow{4}\,\overrightarrow{4}\,\overleftarrow{4}\,\overleftarrow{6}\,\overrightarrow{6}\,\overrightarrow{4}\,\overleftarrow{4}\,\overleftarrow{6}\,\overrightarrow{6}\,\overrightarrow{4}\,\overleftarrow{4}\,\overrightarrow{4}\ \dots$$

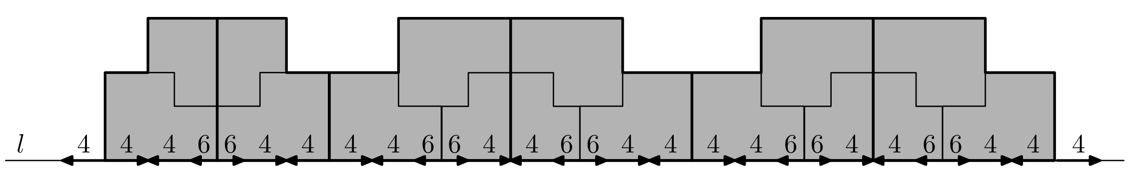
and we want to reconstruct the tiling in the stripe of width d. We apply the inverse map two times and get the shadow

$$\dots\ \overleftarrow{6}\,\overrightarrow{6}\,\overrightarrow{4}\,\overleftarrow{4}\,\overleftarrow{6}\,\overrightarrow{6}\,\overrightarrow{4}\,\overleftarrow{4}\,\overrightarrow{4}\,\overleftarrow{4}\,\overleftarrow{6}\,\overrightarrow{6}\,\overrightarrow{4}\,\overleftarrow{4}\,\overrightarrow{4}\,\overleftarrow{4}\,\overleftarrow{6}\,\overrightarrow{6}\ \dots.$$

Then apply the procedure of Claim 2 to construct the tiling with this shadow:



Then we apply substitution two times and get the sought tiling:



Lemma 2 and Theorem 3(a) are proved. □

4.3.2 Proof of Theorem 3(b)

Let S be an A2 tiling of a half-plane, which is not an infinite supertile. Let l denote the border line of that half-plane. As we have seen, S is a disjoint union of two infinite supertiles S_1, S_2 tiling quadrants. Let r denote the ray that separates those quadrants. Let S_1', S_2' denote the reflections of S_1, S_2 in the axis l.

Then $S_1 \cup S_2 \cup S_1' \cup S_2'$ is an A2 tiling of the entire plane, which is a disjoint union of tilings $S_1 \cup S_1'$ and $S_2 \cup S_2'$ of half-planes separated by the line $r \cup r'$. By Theorem 3(a) the tilings $S_1 \cup S_1'$ and $S_2 \cup S_2'$ are reflections of each other in the axis $r \cup r'$, q.e.d.

4.3.3 Proof of Theorem 3(c).

As we have seen, every A2 tiling of a quadrant is an infinite supertile. Let us show that there are only three such tilings. Let (H, α) be a succinct representation of an infinite supertile tiling a quadrant where H is a large tile.

By Proposition 3 infinite supertiles with succinct representations (H, α) and (G, β) are congruent iff α and β are equivalent (we assume that both H, G are large tiles). Recall that sequences α, β of letters l, s are equivalent if $\alpha = u\gamma$ and $\beta = v\gamma$ for some u, v of the same weighted length; calculating the weighted length we count every letter l with weight 1 and every letter s with weight 2.

By Theorem 1 the tiling with succinct representation (H, α) tiles a quadrant iff α has a tail $slslslsl\dots$. Let us show that there are three non-equivalent sequences α having such tail, namely

$$slslslsl\dots, \qquad lslslslsl\dots, \qquad llslslslsl\dots \tag{1}$$

Indeed, the weighted lengths of the sequences s and ll coincide. Thus replacing in any sequence any letter s by ll we get an equivalent sequence. Vice verse, replacing any block ll by s we get an equivalence sequence. Therefore every sequence with the tail $slslslsl\dots$ is equivalent to a sequence of the form $uslslslsl\dots$ where u is a finite sequence consisting only of l's, $u = ll\dots l$ (we replace each s before the tail by ll). Now replace in u every triple of consecutive l's by sl. The resulting sequence is equivalent to the original one and equals to one of the sequences (1), depending on the residue of the length of u modulo 3.

On the other hand, as $w(sl) = 3$ and $w(\text{empty word})$, $w(l)$, $w(ll)$ are not congruent modulo 3, the three above sequences are pairwise non-equivalent. One can see in Fig. 15 how the corresponding tilings of quadrants look like. The first one is obtained if we put the origin of the quadrant in the bottom left corner. To obtain the second tiling imagine that the origin of the quadrant is in the bottom right corner. For the third consider the top left corner. The substitution transforms these tilings as follows: $1 \to 2 \to 3 \to 1$.

4.4 Proof of Theorem 4

We are given an A2 tiling T of a convex set and have to show that it is self-similar. By Theorem 3, if T is not an infinite supertile (in which case we are done), it consists either of two, or of four axial symmetrical infinite supertiles.

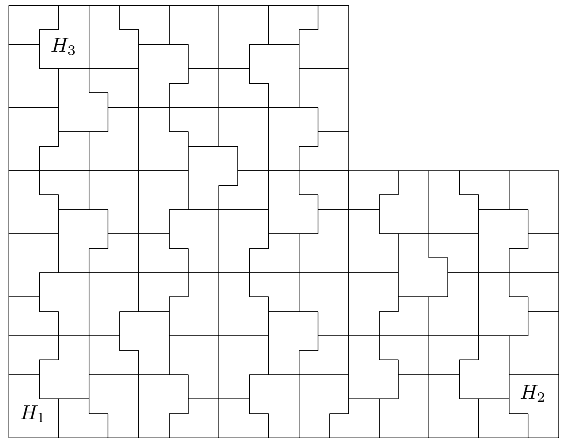


Figure 15: The picture shows three different tilings of the quadrant: they have succinct representation $(H_1, slslsl\dots)$, $(H_2, lslslsl\dots)$ and $(H_3, llslslsl\dots)$.

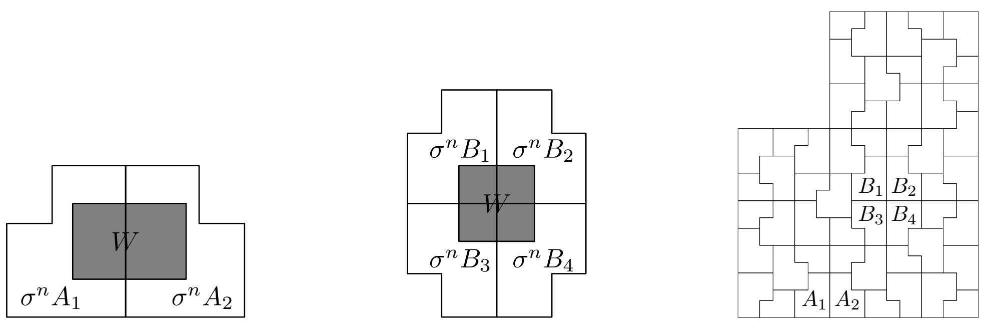


Figure 16: On the left: the pattern W is covered by two symmetrical supertiles $\sigma^n A_1, \sigma^n A_2$. In the middle: the pattern W is covered by four symmetrical supertiles $\sigma^n B_1, \sigma^n B_2, \sigma^n B_3, \sigma^n B_4$. On the right: the supertile of level 8 includes both patterns $\{A_1, A_2\}$ and $\{B_1, B_2, B_3, B_4\}$

Consider the first case: $T = S_1 \sqcup S_2$ where S_1, S_2 are infinite supertiles. Let W be a finite subset of T. We have to show that W is a subset of a supertile. Let $W_i = W \cap S_i$ for $i = 1, 2$. For an integer n, apply n times composition to tilings S_1, S_2. If n is large enough, then the area tiled by W_1 is covered by a single tile A_1 from $\sigma^{-n} S_1$. By Remark 3 on page 14 w.l.o.g. we may assume that A_1 is a large tile and its back lies on the line l separating $[S_1]$ from $[S_2]$. The mirror image A_2 of A_1 belongs to $\sigma^{-n} S_2$. If n is large enough then the area tiled by W_2 is covered by A_2, as shown on Fig. 16 (page 21). It remains to notice the pattern consisting of two tiles sharing the their largest sides is included in the supertile of level 8 (see Fig 16). Applying n substitutions to that supertile we obtain a supertile including W.

Consider now the second case $T = S_1 \sqcup S_2 \sqcup S_3 \sqcup S_4$ where S_1, S_2, S_3, S_4 are axial symmetrical infinite supertiles tiling quadrants. Let W be a finite subset of T and let $W_i = T \cap S_i$ for $i = 1, 2, 3, 4$. Arguing in a similar way we can show that for some n each set $[W_i]$ is covered by a single large tile $B_i \in \sigma^{-n} T$. By Remark 4 (page 16) we may assume that

B_1, B_2, B_3, B_4 are large tiles sharing the their backs and bottoms, as shown on Fig. 16. The pattern consisting of four large tiles shown in Fig. 16 can be found in the supertile of level 8 (Fig. 16). Applying n substitutions to that supertile we obtain a supertile including W.

4.5 Proof of Theorem 5

The tiling T (a finite part of it) is shown on Fig. 17 (page 29). This tiling is defined as follows. Consider a large d-tile F and the point P on its largest side (see Fig. 18 on page 30). Let h denote the homothety with center P and ratio ψ^{-4} and z the mapping $X \mapsto \sigma^4(h(X))$ on d-tilings. Consider the supertile $z(\{F\})$ (the supertile S on Fig. 18). It is easy to verify that it indeed contains the tile F. This implies that

$$\{F\} \subset z(\{F\}) \subset z(z(\{F\})) \subset \dots$$

Let T_{up} be the union of this chain of tilings:

$$T_{up} = \{F\} \cup z(\{F\}) \cup z(z(\{F\})) \cup \dots.$$

One can see that the tiling T_{up} is an infinite supertile with succinct representation $(F, ssss\dots)$.

Now consider the rotation $\mathcal{R}$ by 180° around the point P and let

$$T_{bottom} = \mathcal{R}(T_{up}) \text{ and } T = T_{up} \cup T_{bottom}.$$

It is easy to see that both T_{up} and T_{bottom} are fixed points of the mapping z.

The theorem follows from two claims:

Claim 1. *The tiling T satisfies the Arrow rule everywhere except for sides shared by pairs of tiles $(E, \mathcal{R}(F))$, $(F, \mathcal{R}(F))$ and $(F, \mathcal{R}(E))$.*

Claim 2. *If T' is an infinitely composable tiling of the plane that includes almost all tiles from T, then $T' = T$.*

Proof of Claim 1. If two adjacent tiles from T both belong to T_{up} or both belong to T_{bottom}, then they satisfy the Arrow rule, as both T_{up} and T_{bottom} are infinite supertiles. Therefore it remains to verify the Arrow rule is met for all tiles from $T \setminus \{F, \mathcal{R}(F)\}$ that are adjacent to the line l separating T_{up} from T_{bottom} (see Fig. 17 on page 29). For the pairs $(D, \mathcal{R}(G))$, $(E, \mathcal{R}(G))$, $(G, \mathcal{R}(E))$, $(G, \mathcal{R}(D))$ this verification can be done by hand.

For the remaining tiles we can argue as follows. Let V denote the set of all tiles from T_{up} that are adjacent to l and are not marked grey on Fig. 17, that is,

$$V = \{D, E, F, G\}.$$

Let W denote the remaining tiles from T_{up} that have names (they all are marked grey), that is,

$$W = \{A, B, C, H, I, J\}.$$

The claim follows from the following two facts that can be verified by hand:
Fact 1. $z(V) = V \cup W \cup U$, where U is a set of tiles that all do not touch the line l separating T_{up} from T_{bottom} (we can find U explicitly, however for our argument we need only that all tiles from U do not touch the line l).
Fact 2. Let $\mathcal{S}$ denote the reflection in the axis l. Then $\mathcal{R}(W) = \mathcal{S}(W)$. More specifically, $\mathcal{R}(A) = \mathcal{S}(J)$, $\mathcal{R}(B) = \mathcal{S}(I)$ and so on.

The definition of T_{up} and the first fact imply that

$$T_{up} = V \cup \bigcup_{i=0}^{\infty} z^i(W) \cup \bigcup_{i=0}^{\infty} z^i(U).$$

Indeed, it is easy to show by induction on n that

$$z^n(\{F\}) \subset V \cup \bigcup_{i=0}^{n} z^i(W) \cup \bigcup_{i=0}^{n} z^i(U)$$

for all n. Conversely, it is easy to see that $V \subset z^2(\{F\})$ hence $V \cup W \cup U = z(V) \subset z^3(\{F\})$, which implies that

$$z^i(W) \cup z^i(U) \subset z^{i+3}(\{F\}) \subset T_{up}$$

for all i.

Similarly,

$$T_{bottom} = \mathcal{R}(V) \cup \bigcup_{i=0}^{\infty} z^i(\mathcal{R}(W)) \cup \bigcup_{i=0}^{\infty} z^i(\mathcal{R}(U)).$$

These representations of T_{up} and T_{bottom} and the second fact imply that all tiles from $T_{up} \setminus V$ that are adjacent to the line l are mirror images of tiles from $T_{bottom} \setminus \mathcal{R}(V)$. Hence the Arrow rule is met for those tiles. □

Proof of Claim 2. Consider the chain of tilings $S_0 \subset S_1 \subset S_2 \subset \dots$ where $S_0 = \{F, \mathcal{R}(F)\}$ and $S_{i+1} = z(S_i)$ (recall that $z(S) = \sigma^4 h(S)$). The set $S_1 = z(S_0)$ is marked grey on Fig. 19 (page 31).

The claim holds, since $T = \bigcup_{i=0}^{\infty} S_i$ and T is a fixed point of z. More specifically, let T' be an infinitely composable tiling of the plane that includes almost all tiles from T. Since $T = \bigcup_{i=0}^{\infty} S_i$, for some i we have

$$T \setminus S_i \subset T'. \tag{2}$$

Assume first that $i = 0$. It is easy to verify by hand that T is the only tiling of the plane that includes $T \setminus S_0$ and hence $T' = T$.

Assume that $i > 0$. We claim that in this case we have

$$T \setminus S_{i-1} \subset z^{-1}(T'). \tag{3}$$

To prove this claim, apply σ^{-4} to the inclusion (2). We obtain

$$\sigma^{-4}(T \setminus S_i) \subset \sigma^{-4} T'$$

Since T is the disjoint union of S_i and $T \setminus S_i$ we have

$$\sigma^{-4}T = \sigma^{-4}S_i \sqcup \sigma^{-4}(T \setminus S_i)$$

and hence

$$\sigma^{-4}(T \setminus S_i) = \sigma^{-4}T \setminus \sigma^{-4}S_i.$$

Thus

$$\sigma^{-4}T \setminus \sigma^{-4}S_i \subset \sigma^{-4}T'.$$

Applying h^{-1} to this inclusion we get (3), since T is a fixed point of z.

In the above arguments we implicitly used Lemma 1 several times. To to show that we may do that, we have to prove that all the tilings of the form

$$\sigma^{-j}S_i, \qquad \sigma^{-j}(T \setminus S_i), \qquad j = 0, 1, 2, 3$$

are proper. These tilings may be obtained from tilings S_0 and $T \setminus S_0$ by applying substitution $4i - j$ times and then applying a homothety. If $4i - j \geqslant 2$, then the tilings are proper, since the can be obtained from some tilings by applying σ^2. In the remaining case $i = 1, j = 3$ we can verify by hand that the tilings $\sigma^{-3}S_1$ and $\sigma^{-3}(T \setminus S_1)$ are proper.

Repeating this trick i times we can show that $T \setminus S_0 \subset z^{-i}(T')$. As we have seen, this implies $z^{-i}(T') = T$ and hence $T' = T$, as T is a fixed point of z. □

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A Appendix

A.1 Proof of Lemma 1

(a) Every small tile F from S produces the same tile ($F \cup$ the brother of F) in $\sigma^{-1}S$ and $\sigma^{-1}T$. Every large tile G from S produces the same tile ($F \cup$ the sister of G) in $\sigma^{-1}S$ and $\sigma^{-1}T$, if the sister of G is in S (and hence in T). Otherwise S (and hence T) contains a tile H that covers the cavity of G. This tile is different from the sister of G. Thus in this case the tile G produces itself in both $\sigma^{-1}S$ and $\sigma^{-1}T$.

(b) By item (a) both tilings $\sigma^{-1}S$ and $\sigma^{-1}T$ are subsets of $\sigma^{-1}(S \cup T)$. Hence $\sigma^{-1}S \cup \sigma^{-1}T \subset \sigma^{-1}(S \cup T)$. This inclusion cannot be proper, as $\sigma^{-1}S \cup \sigma^{-1}T$ and $\sigma^{-1}(S \cup T)$ tile the same set.

A.2 Proof of Proposition 1

(a) By induction: we will show that if a tiling T is A2 then so is σT. Assume that T is an A2 d-tiling. Color in the tiling T all large and small tiles as shown on Fig. 4(b,c) (page 4). Such coloring will be called *canonical.* Assume that in the canonical coloring of T the colors and orientations in all pairs of adjacent tiles match. We have to show that the same holds for σT.

To verify this, cut all large d-tiles from T, as shown on Fig. 4(a). Then change the colored segments in the original canonical coloring using the following substitution:

$$\overrightarrow{6} \to \overrightarrow{5}$$
$$\overrightarrow{5} \to \overrightarrow{4}$$
$$\overrightarrow{4} \to \overrightarrow{3}$$
$$\overrightarrow{3} \to \overleftarrow{4}\,\overleftarrow{6}.$$

The reverse arrows in $\overleftarrow{4}$ and $\overleftarrow{6}$ mean that we reverse orientation. After that color the cut by colors $\overrightarrow{4}$, $\overrightarrow{5}$, $\overrightarrow{6}$, as shown in Fig. 4(a).

As the transformation of colors of every segment does not depend on the tile it belongs to, it does not destroy the matching requirement. Therefore it remains to verify that after transformation we get the canonical coloring of σT. This can be verified just by comparing Fig. 4(a), 4(b) and 4(c): the transformation of colors applied to Fig. 4(c) produces the coloring as in Fig. 4(b) and the transformation of colors applied to Fig. 4(b) produces the coloring as in Fig. 4(a).

(b) Let T be an A2 tiling of a convex set. We have to show that for every small tile $H \in T$ there is a large G located as shown on Fig. 4(a). Indeed, the cavity in H formed by arrows $\overrightarrow{5}, \overrightarrow{6}$ is somehow filled by another tile G in T. Notice that only large tiles have a right angle with arrows $\overrightarrow{5}, \overrightarrow{6}$ thus G is a large tile. There is only one such angle in every large tile and only one way to properly attach a large tile to a small tile to fill the gap, namely the way shown in Fig. 4(a).

(c) Let T be an A2 tiling of a convex set. We have to show that $\sigma^{-1}T$ is again an A2 tiling. This is done in a way similar to that in the proof of item (a).

The tiling $\sigma^{-1}T$ and its canonical coloring can be obtained from the canonical coloring of T in two steps: erase sides $\overleftarrow{4}, \overleftarrow{5}, \overleftarrow{6}$ shared by each pair (sister, brother), to get the tiling $\sigma^{-1}T$ and then replace the colors using the map:

$$\begin{aligned} \overrightarrow{6}\,\overrightarrow{4} &\to \overleftarrow{3} \\ \overrightarrow{5} &\to \overrightarrow{6} \\ \overrightarrow{4} &\to \overrightarrow{5} \\ \overrightarrow{3} &\to \overrightarrow{4}. \end{aligned}$$

Note that, after erasing sides shared by each pair of small and large tile, every occurrence of arrow $\overrightarrow{6}$ in a tile from T is followed by an arrow $\overrightarrow{4}$ belonging to the same side of the same tile. This is easy to verify looking at Fig. 4(b,c) (the arrow $\overrightarrow{6}$ has two occurrences on sides of the large tile and two occurrences on sides of the small tile; all they are followed by $\overrightarrow{4}$ except for the side of a small tile that belongs to the cavity—but such sides have been erased). An arrow $\overrightarrow{4}$ is replaced using the first rule, if it follows $\overrightarrow{6}$, and using the third rule otherwise. In the obtained coloring the colors and orientations match, as the transformation of every arrow does not depend on the tile whose side it belongs to.

It remains to verify that the resulting coloring of $\sigma^{-1}T$ is indeed canonical. This can be verified by comparing Fig. 4(a), Fig. 4(b) and Fig. 4(c), as in the proof of item (a).

A.3 Proof of Proposition 2

Items (a) and (c) follow from the following

Claim. *Assume that T is a finite d-supertile or an infinitely composable d-tiling. Assume that H is any of its tiles (small or large). Then there is a unique sequence $H_0 \subset H_1 \subset H_2 \subset \dots$ that starts with H and ends with $[T]$, if T is a finite supertile, and is infinite, if T is infinitely composable, and such that H_i is either the son, or the daughter of H_{i+1} for all i, and $S_d(H_i) \subset T$ for all i.*

Proof. For finite supertiles the statement can be proved by induction on the level of T. If T is a supertile of level -1 or 0, then the statement is obvious. Otherwise T is a disjoint union of supertiles T' and T'' of smaller levels, which tile the son and the daughter of T, respectively. Since T' and T'' are disjoint, we have either $H \in T'$, or $H \in T''$. In the first case the last but one tile in the sought sequence $H_0, H_1, H_2, \dots$ must be $[T']$ and the statement for T follows from the induction hypothesis for T'. Similarly, in the second case the statement for T follows from the induction hypothesis for T''.

For infinitely composable tilings we are unable to use similar arguments, since the sequence $H_0, H_1, H_2, \dots$ must be infinite. Let us first prove that such sequence exists. For every l consider the tiling $\sigma^{-l}T$ obtained from T by l compositions. The tiling T is a disjoint union of supertiles $S_d(G)$ where $G \in \sigma^{-l}T$. Let H_l denote the (unique) tile from $\sigma^{-l}T$ such that the supertile $S_d(G)$ contains H.

We claim that for all l the tile H_l is either the son, or the daughter of H_{l+1}, or H_l coincides with H_{l+1}. Indeed, the tiling $\sigma^{-l}T$ is the decomposition of the tiling $\sigma^{-l-1}T$. If H_{l+1} is a small tile in the tiling $\sigma^{-l-1}T$, then its decomposition coincides with it, thus H_{l+1} is in $\sigma^{-l}T$ and hence H_l and H_{l+1} coincide. Otherwise the tiling $\sigma^{-l}T$ contains the result of decomposition of H_{l+1}, that is, the son F and the daughter G of H_{l+1}. Since

$$H \in S_d(H_{l+1}) = S_d(F) \sqcup S_d(G),$$

the tile H belongs either to $S_d(F)$, or to $S_d(G)$. In the first case H_l must be equal to F and otherwise $H_l = G$.

Removing repetitions from the sequence $H_0, H_1, H_2, \dots$ we obtain the sought sequence of tiles.

Let us prove now that that such chain is unique. Assume that there are two such chains $H_0 \subset H_1 \subset H_2 \subset \dots$ and $G_0 \subset G_1 \subset G_2 \subset \dots$. Let us show by induction on n that $H_n = G_n$. By assumption we have $H_0 = G_0 = H$.

Induction step: assume that $H_n = G_n$. By way of contradiction, assume that $H_{n+1} \neq G_{n+1}$. Then the tile $H_n = G_n$ is the son of H_{n+1} and the daughter of G_{n+1} (or the other way around, but the other case in entirely similar). Let l stand for the level of the supertile $S_d(H_n) = S_d(G_n)$. The levels of supertiles $S_d(H_{n+1})$ and $S_d(G_{n+1})$ are $l+1$ and $l+2$ respectively. By Lemma 1(a) both tilings $\sigma^{-l}S_d(H_{n+1})$, $\sigma^{-l}S_d(G_{n+1})$ are included into $\sigma^{-l}T$. The first tiling consists of H_n and its sister. The second one consists of H_n and the son and the daughter of the brother of H_n. Hence the tiling $\sigma^{-l}T$ contains the sister of H_n and the son of the brother of H_n, which overlap (see Fig. 20 on page 32). The obtained contradiction proves that $H_{n+1} = G_{n+1}$. □

(a) Let T be an infinite supertile with representation $H_0 \subset H_1 \subset H_2 \subset \dots$ and H any of its tiles. Consider any n such that $H \in S_d(H_n)$. By the claim there is a chain of tiles $G_0 \subset G_1 \subset \dots \subset G_l$ such that $G_0 = H$, $G_l = H_n$ and G_i is either the son, or the daughter of G_{i+1} for all i. Then the sequence

$$G_0, G_1, \dots G_l, H_{n+1}, H_{n+2} \dots$$

is the sought representation of T.

The uniqueness part of the claim implies that such representation is unique.

(b) Let F be any tile from T. Then there are k, l such that $F \in S_d(H_k)$ and $F \in S_d(G_l)$. Let $I_0, I_1, \dots, I_a$ denote the representation of $S_d(H_k)$ that starts with F and $J_0, J_1, \dots, J_b$ the representation of $S_d(G_l)$ that starts with F. Then both sequences of tiles

$$I_0, I_1, \dots, I_a, H_{k+1}, H_{k+2}, \dots \qquad J_0, J_1, \dots, J_b, G_{l+1}, G_{l+2}, \dots$$

are representations of T that both start with F. By item (a) these sequences coincide. Note that H_{k+i} is the $(a+i)$th terms in the first representation and G_{l+j} is the $(b+j)$th in the second representation (for all i, j). Hence $H_{b+k+i} = G_{a+l+i}$ for all i.

(c) Consider the chain $H_0 \subset H_1 \subset H_2 \subset \dots$ existing by the claim. Then the union $\cup_{n=1}^{\infty} S_d(H_n)$ is the sought supertile. To prove uniqueness, notice that the representation of any supertile S satisfying the statement and starting with H must satisfy the claim.

A.4 Proof of Proposition 3

'If' part. Assume that $\alpha = u\gamma$ and $\beta = v\gamma$ where $w(u) = w(v)$. Then the supertiles with succinct representations (H, u) and (G, v) have the same level (equal to $w(u) = w(v)$) and hence are congruent. This implies that the infinite supertiles with succinct representations $(H, u\gamma)$ and $(G, v\gamma)$ are congruent as well.

'Only if' part. We are given congruent infinite supertiles T, S with succinct representations (H, α) and (G, β), respectively. W.l.o.g. we may assume that the tilings T, S coincide (otherwise we apply to H the isometry f that maps T to S and obtain another succinct representation $(f(H), \alpha)$ of the supertile S).

Thus we are given two different succinct representations (H, α) and (G, β) of the same infinite supertile S. By Proposition 2(b) the corresponding representations $(H = H_0), H_1, H_2, \dots$ and $(G = G_0), G_1, G_2, \dots$ have the same tail, that is, $H_{n+i} = G_{m+i}$ for some m, n and all $i \geqslant 0$. Thus $\alpha = u\gamma$ and $\beta = v\gamma$ where (H, u) and (G, v) are succinct representations of the d-supertile $S_d(H_n) = S_d(G_m)$, and γ is the infinite s-l-sequence with

$$\gamma_i = \begin{cases} s, & \text{if } H_{n+i} = G_{m+i} \text{ is the daughter of } H_{n+i+1} = G_{m+i+1}, \\ l & \text{otherwise.} \end{cases}$$

We have $w(u) = w(v)$, since (H, u) and (G, v) are succinct representations of the same supertile.

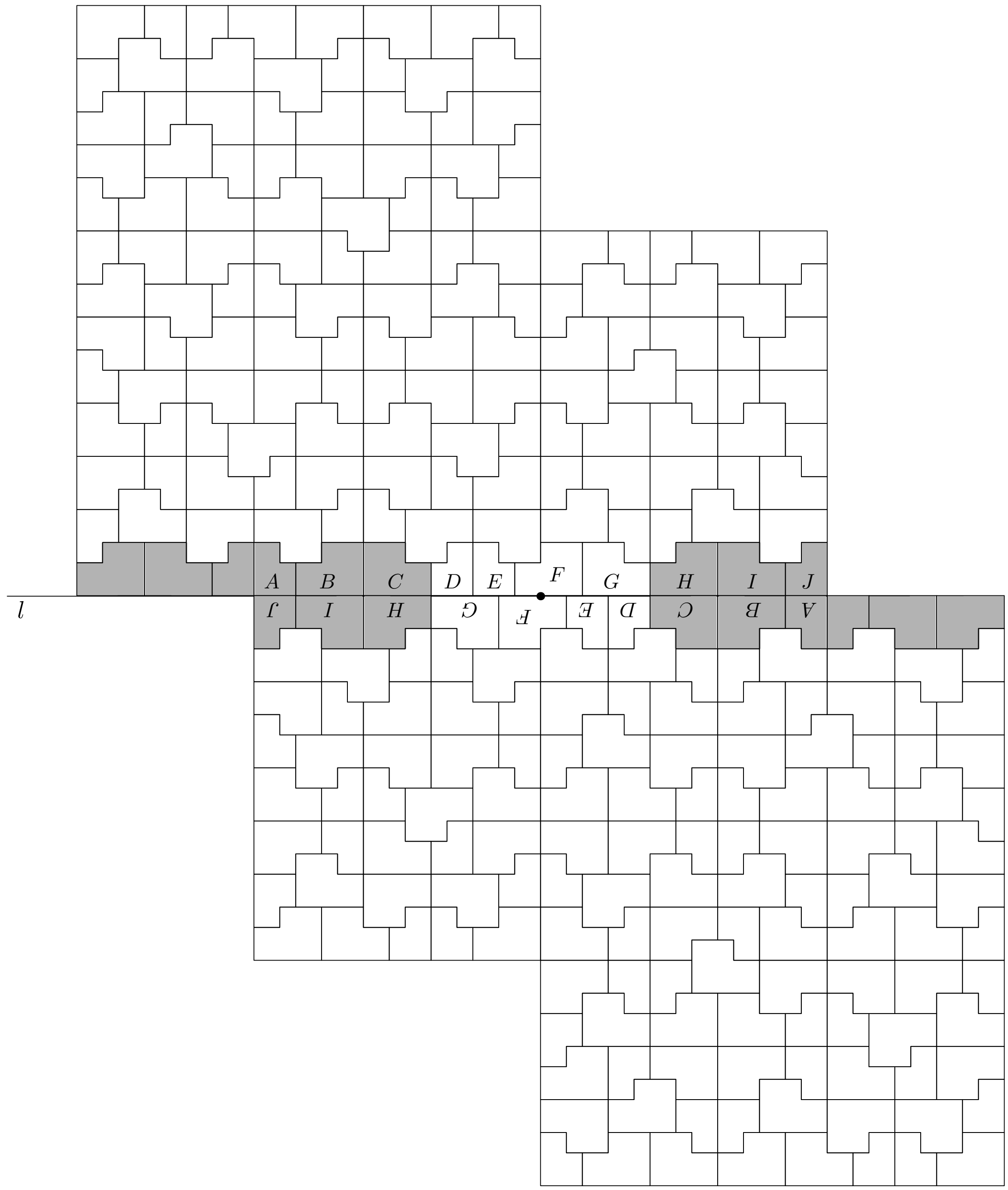

Figure 17: The tiling T is the union of tilings T_{up} and T_{bottom}. The tiling T_{bottom} is obtained from T_{up} by rotation by 180° around a point on the line l. Some tiles from T have names.

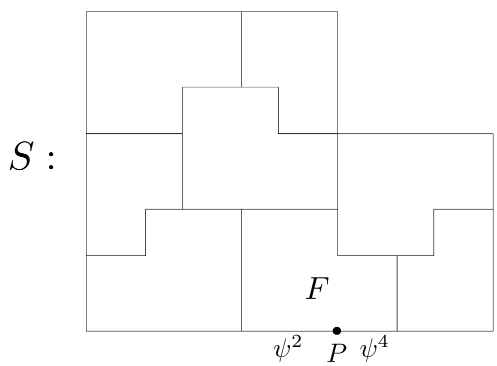

Figure 18: The picture shows a d-supertile S of level 4. It contains a large d-tile F. The supertile S can be obtained from F by applying the homothety with center P and then applying four substitutions. The point P divides the largest side of F (of length d) into segments of lengths $d\psi^2, d\psi^4$.

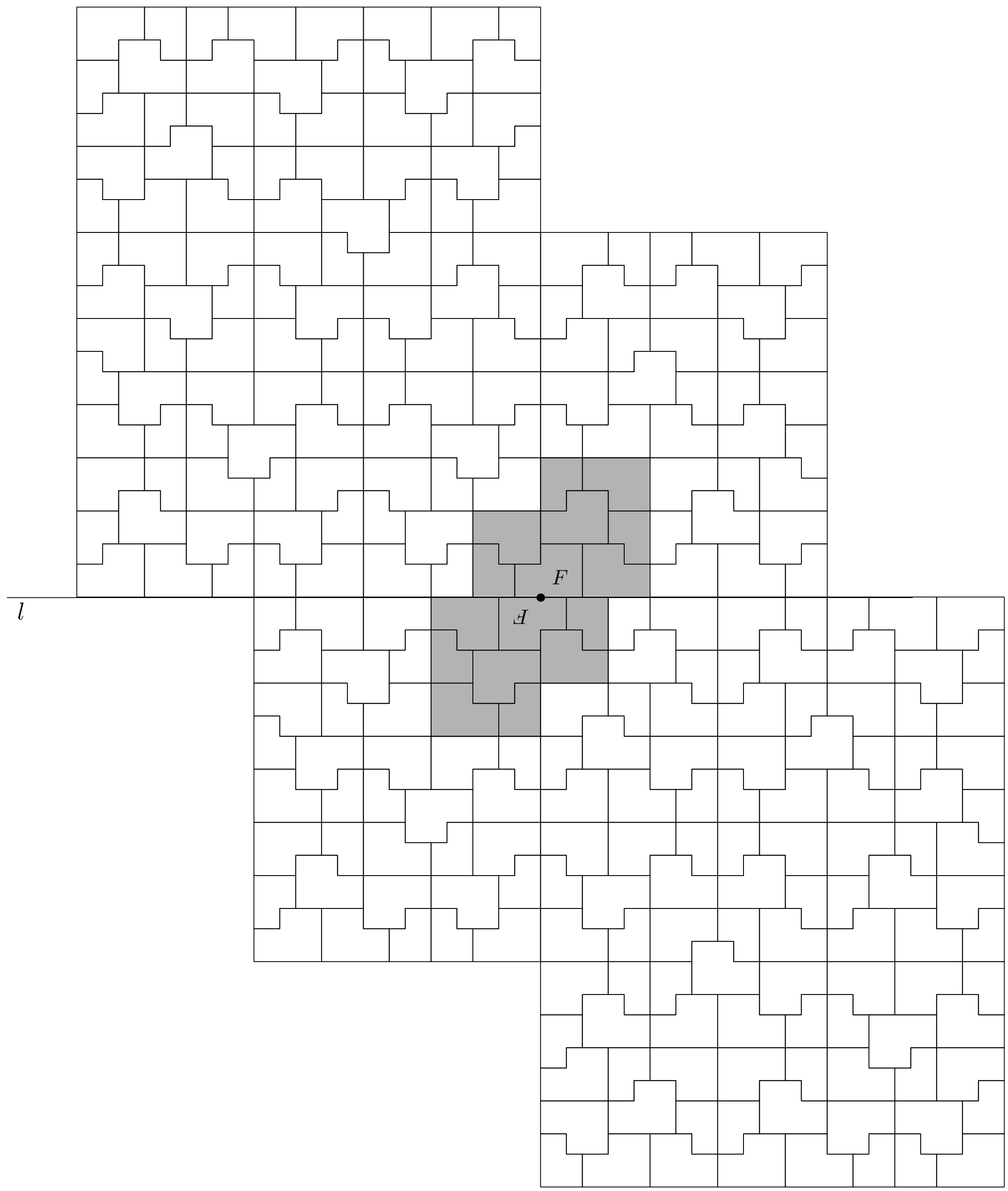

Figure 19: The tiling T includes the set $S_1 = z(\{F, \mathcal{R}(F)\})$ marked grey.

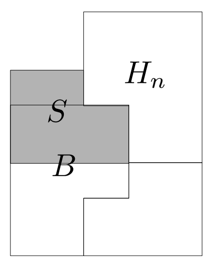

Figure 20: The sister S (marked grey) of the large tile H_n overlaps with the son B of the brother of H_n.