The chromatic number of space with Chebyshev metric without long monochromatic unit arithmetic progressions

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For an $n$-dimensional normed space $\mathbb{R}^n$, the chromatic number $\chi(\mathbb{R}^n)$ is the smallest $r$ such that there exists a coloring of the points of $\mathbb{R}^n$ with $r$ colors, i.e. an $r$-coloring, and with no two points of the same color unit distance apart. A general result by Kupavskii [3] establishes an upper bound on this quantity depending only on the dimension: $\chi(\mathbb{R}^n) \leq (4 + o(1))^n$ as $n \to \infty$.

A subset $M' \subset \mathbb{R}^n$ is called an $N$-isometric copy of $M$ if there exists a bijection $f: M \to M'$ such that $\|x - y\|_N = \|f(x) - f(y)\|_N$ for all $x, y \in M$. Given a normed space $\mathbb{R}^n$ and a subset $M \subset \mathbb{R}^n$, the chromatic number $\chi(\mathbb{R}^n, M)$ is the smallest $r$ such that there exists an $r$-coloring of $\mathbb{R}^n$ with no monochromatic $N$-isometric copy of $M$. In these terms, $\chi(\mathbb{R}^n) = \chi(\mathbb{R}^n, I)$, where $I$ is a two-point set.

We consider a sequence of positive reals $\lambda_1, \ldots, \lambda_k$, given in [4]. We call a set $\{0, \lambda_1, \lambda_1 + \lambda_2, \ldots, \sum_{t=1}^k \lambda_t\} \subset \mathbb{R}$ a baton and denote it by $B(\lambda_1, \ldots, \lambda_k)$. In case $\lambda_1 = \cdots = \lambda_k = 1$, i.e., if the set is just a unit arithmetic progression, we simply denote it by $B_k$ for a shorthand. Erdős et al. [5] showed that any baton $B$ of at least three points is not Ramsey, since $\chi(\mathbb{R}^2, B) \leq 16$ for all $n \in \mathbb{N}$. Moreover, they proved that

$$\chi(\mathbb{R}^2, B_k) = 2 \quad (1)$$

for all $k \geq 5$ and for all natural $n$. (Note that it is unknown [1] whether the values 5 and 16 here are tight.)

In a recent series of papers [4, 6, 7], the authors studied the chromatic numbers $\chi(\mathbb{R}^n, M)$ for the $n$-dimensional Chebyshev spaces $\mathbb{R}_\infty^n$. Among the other results, they proved that

$$\chi(\mathbb{R}^n_\infty, B_k) \geq \left(\frac{k + 1}{k}\right)^n \quad (2)$$

for all $k, n \in \mathbb{N}$. This inequality shows that, unlike the Euclidean case, for any given $k$, every two-coloring of $\mathbb{R}^n$ contains a monochromatic $\ell_\infty$-isometric copy of $B_k$ whenever the dimension $n$ is large enough in terms of $k$. The next theorem shows that this is not that case in the ‘opposite’ setting, when $k$ is sufficiently large in terms of $n$.

**Theorem 1.** Given $n \in \mathbb{N}$, there exists a two-coloring of $\mathbb{R}^n$ with no monochromatic $\ell_\infty$-isometric copies of all batons $B(\lambda_1, \ldots, \lambda_k)$ such that $\max_i \lambda_i \leq 1$ and $\sum_{t=1}^k \lambda_t \geq 5^n$. In particular, for all $n \in \mathbb{N}$ and $k \geq 5^n$, we have $\chi(\mathbb{R}^n_\infty, B_k) = 2$.

For the prove of the theorem we introduced the next notation: a *snake hypersurface* $S^n(a_1, b_1, \ldots, a_n, b_n)$, where $n \in \mathbb{N}$, is an $n$-dimensional piecewise linear hypersurface in $\mathbb{R}^{n+1}$ that depends on $2n$ positive real parameters. The definition is by induction on $n$. Let $S^0(\emptyset) := \{0\}$ be just the origin of the
line. For $n > 0$, we define $S^n(a_1, b_1, \ldots, a_n, b_n)$ by

$$S^n(a_1, b_1, \ldots, a_n, b_n) = S^{n-1}(a_1, b_1, \ldots, a_{n-1}, b_{n-1})$$

$$+ \mathbb{Z} \cdot \{a_n \cdot e_{n+1} - b_n \cdot 1_n\} + [0, a_n) \cdot e_{n+1} \cup (0, b_n] \cdot 1_n,$$

(3)

where $e_1, \ldots, e_n$ be the standard basis vectors for $\mathbb{R}^n$, and $1_n \in \mathbb{R}^n$ be their sum, i.e., a vector with all $n$ coordinates being unit.

References


