On Neighbourhood Semantics Thirty Years Later

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1 Introduction

The research of Dov Gabbay includes different parts of logic, and in many cases it essentially influenced further development. Problems considered here are motivated by two his papers, [Gabbay, 1975] and [Gabbay and de Jongh, 1974]. These papers appeared at the beginning of the 1970s, at the time of remarkable events in modal logic, when all of a sudden the whole area was found full of difficult problems and nice theorems, like flowers growing in high mountains, and young researchers came for new interesting discoveries.

General problems in neighbourhood semantics were first addressed by D. Gabbay and M. Gerson in [Gabbay, 1975; Gerson, 1975a; 1975b]. At that time the interest in neighbourhood semantics was rather moderate, but now it is clear that neighbourhood approach may be quite useful in different kinds of modal logic: spatial, epistemic, conditional etc. [van Benthem and Sarenac, 2004; Aiello, 2002].

This paper studies neighbourhood completeness and compactness for modal and intermediate logics. We show that completeness and compactness are closely related: noncompact logics (in Thomason's sense [Thomason, 1972]) may be helpful for distinguishing Kripke and neighbourhood completeness, which is the problem studied in [Gabbay, 1975]. Three technical details are crucial here: K. Fine's frame [Fine, 1974], the axiomatisation of binary finite trees [Gabbay and de Jongh, 1974], and the ultrabouqet construction [Shehtman, 1998].

Some of the results presented here were published, but only in Russian, in a hardly available paper [Shehtman, 1980] and later in the author's Thesis [Shehtman, 2000] (even less available). So we give a slightly modified exposition of these results.

The plan of the paper is as follows. Section 2 contains very basic material on Kripke and neigbourhood semantics; but some notions (such as different kinds of compactness in modal logic) are not widely known. In Section 3 we recall properties of ultrabouqets of topological spaces proved in [Shehtman, 1998]. Ultrabouqets of neighbourhood **K4**-frames are considered in Section 4. The latter construction is new, and it is used only in Section 9, so the readers can skip it if they are interested in other parts of the paper. Section 5 presents an example of a TKN-noncompact extension of **Grz** from [Shehtman, 1980]. Basing on it, we construct a new rather simple example of a relatively incomplete modal logic in Section 6. The same is done for intermediate logics: Section 7 contains an example of TK-noncompactness from [Shehtman, 1980], and Section 8 — a new example of relative incompleteness. Note that the earlier example of a relatively incomplete intermediate logic is quite complicated. This example is recalled in Section 10, but without the laborious proof (given in full detail in [Shehtman, 2000]). Section 9 also proves new results: N-compactness for all extensions of **GL** and **Grz**. Section 10 contains some hints for further results and some questions.

2 Preliminaries

The material of this Section is rather standard, most of it can be found in the first chapters of [Chagrov and Zakharyaschev, 1997], but our notation is slightly different.

In this paper we consider monomodal and intermediate propositional logics. So *modal formulas* are constructed from the countable set of propositional variables $PV = \{p, q, \dots\}$, the constant $\bot$, and the connectives $\to$ $\wedge$, $\vee$, $\Box$; the derived connectives are: $\neg$, $\top$, $\leftrightarrow$, $\Diamond$. *Intuitionistic formulas* are modal formulas without occurrences of $\Box$.

A *modal logic* is a set of modal formulas containing all classical tautologies, the axiom $\Box(p \to q) \to (\Box p \to \Box q)$ and closed under Modus Ponens, $\Box$-introduction, and Substitution; we consider only consistent logics (i.e. not containing $\bot$).

An *intermediate logic* is a consistent set of intuitionistic formulas closed under (intuitionistic) Substitutions and Modus Ponens and containing the standard axioms of Heyting Calculus [Chagrov and Zakharyaschev, 1997].

The minimal modal logic is denoted as usual by **K**. Intuitionistic logic (denoted by **H**) is the smallest intermediate logic.

For a logic $\mathbf{\Lambda}$ and a formula A, the notation $\mathbf{\Lambda} \vdash A$ is used as an equivalent to $A \in \mathbf{\Lambda}$. For a set of formulas Γ and a modal (or intermediate) logic $\mathbf{\Lambda}$, the minimal modal (resp., intermediate) logic containing $\mathbf{\Lambda} \cup \Gamma$ is denoted by $\mathbf{\Lambda} + \Gamma$.

Some particular modal logics used in this paper are

$$\begin{aligned}&\mathbf{K4} = \mathbf{K} + \Box p \to \Box\Box p,\\ &\mathbf{S4} = \mathbf{K4} + \Box p \to p,\\ &\mathbf{Grz} = \mathbf{S4} + AG \text{ (Grzegorczyk logic)},\\ &\mathbf{GL} = \mathbf{K4} + AL \text{ (Löb logic)},\end{aligned}$$

where

$$AG := \neg(p \wedge \Box(p \to \Diamond(\neg p \wedge \Diamond p))),$$
$$AL := \Box(\Box p \to p) \to \Box p.$$

Note that the AG is usually written in a different (equivalent) form: as $\Box(\Box(p \to \Box p) \to p) \to p$.

The intermediate logic of our special interest is Gabbay – De Jongh's $\mathbf{H} + Br_2$, where

$$Br_2 := (\bigwedge_{i=0}^{n}(P_i \to \bigvee_{j\neq i} P_j) \to \bigvee_{j\neq i} P_j) \to \bigvee_{i=0}^{n} P_i,$$

and $P_0 := p,\ P_1 := q,\ P_2 := (p\leftrightarrow q)$.

This logic was introduced in [Gabbay and de Jongh, 1974], with the axiom Br_2 in an equivalent form, where P_i are just propositional variables. The above form (proposed by S.K. Sobolev) uses only two variables.

A *neighbourhood frame* is a pair $\mathcal{X} = (X, \Box)$ consisting of a non-empty set with an operation on its subsets $\Box : 2^X \longrightarrow 2^X$. In this paper we consider only **K4**-*frames*, i.e. those, in which $\Box(V_1 \cap V_2) = \Box V_1 \cap \Box V_2$, $\Box X = X$, $\Box V \subseteq \Box\Box V$ and **S4**-*frames* (also satisfying $\Box V \subseteq V$). The latter are nothing but topological spaces ($\Box$ is the interior operator). $\Diamond V = -\Box(-V)$[1] is the closure operator in topological spaces: A *(neighbourhood) model* over $\mathcal{X}$ is a pair $M = (\mathcal{X}, \theta)$, where $\theta : PV \longrightarrow 2^X$ is a *valuation* in $\mathcal{X}$.

The map θ extends to all formulas in the well-known way:
$\theta(\bot) = \varnothing,\ \theta(A \to B) = -\theta(A) \cup \theta(B),\ \theta(A \wedge B) = \theta(A) \cap \theta(B),$
$\theta(A \vee B) = \theta(A) \cup \theta(B),\ \theta(\Box A) = \Box\theta(A).$
The notation $M, x \vDash A$ means $x \in \theta(A)$, which is also read as "a modal formula A is *true at world* x of M". A is called

- *true in* M (notation: $M \vDash A$) if A is true at all worlds of M;
- *valid at* $\mathcal{X}, x$ (notation: $F, x \vDash A$) if A is true at world x under all valuations;
- *valid in* $\mathcal{X}$ (notation: $\mathcal{X} \vDash A$) if A is valid at all worlds of $\mathcal{X}$.

[1] $(-V)$ denotes the compement of V.

$\mathbf{ML}(\mathcal{X})$ denotes the *modal logic of* $\mathcal{X}$, i.e., the set of all modal formulas valid in $\mathcal{X}$. If $\mathbf{\Lambda} \subseteq \mathbf{ML}(\mathcal{X})$, then $\mathcal{X}$ is called a $\mathbf{\Lambda}$*-frame*. For a class of frames $\mathcal{C}$, we denote $\mathbf{ML}(\mathcal{C}) := \bigcap\{\mathbf{L}(\mathcal{X}) \mid \mathcal{X} \in \mathcal{C}\}$ (the *modal logic of* $\mathcal{C}$, or the *modal logic determined by* $\mathcal{C}$). A modal logic determined by some class of neighbourhood frames is called *neighbourhood complete*, or *N-complete.*

A *Kripke frame* is a pair $F = (W, R)$ consisting of a non-empty set with a binary relation. F is associated with the neighbourhood frame $N(F) = (W, \Box)$ such that $\Box V = \{x \in W \mid R(x) \subseteq V\}$. In this paper all Kripke frames are transitive.

A *(Kripke) model* over F is a pair $M = (F, \theta)$, where $\theta : PV \longrightarrow 2^W$ is a valuation; so we can consider M as a neghbourhood model over $N(F)$. For Kripke frames we also use the notations $M, x \vDash A$, $M \vDash A$, $F \vDash A$, $F, x \vDash A$, $\mathbf{ML}(F)$ and the corresponding terminology, as explained above. So transitive Kripke frames are exactly **K4**-frames; reflexive transitive Kripke frames (quasi-ordered sets) are exactly **S4**-frames. Logics determined by classes of Kripke frames are called *Kripke complete* (or *K-complete*).

For semantics of intuitionistic formulas we need topological spaces rather than arbitrary neighbourhood frames. Every intuitionistic formula A translates as a modal formula A^T (by putting $\Box$ in front of every its subformula). A valuation θ in a space $\mathcal{X}$ is called *intuitionistic* if $\theta(s)$ is open for any propositional variable s; then $(\mathcal{X}, \theta)$ is an *intuitionistic topological model.* For any intuitionistic formula A we put:

$$\theta^I(A) := \theta(A^T).$$

Thus we obtain the *intuitionistic extension* θ^I of θ.

We shall also use some special notation. For intutuionistic formulas A, B we put

$$\theta^\bullet(A \to B) := \theta(A, B) := \theta^I(A) - \theta^I(B).$$

Thus $x \notin \theta^I(A \to B)$ iff $x \in \Diamond\theta^\bullet(A \to B)$, and $\theta^\bullet(\neg A) = \theta^I(A)$.

The intuitionistic notions of truth and validity are defined similarly to the modal case:

$M, x \Vdash A$ means $x \in \theta^I(A)$ and is also read as "A is *intuitionistically true at world* x of M", or "M, x *forces* A". A is called

- *intuitionistically true in* M (notation: $M \Vdash A$) if A is intuitionistically true at all worlds of M;
- *intuitionistically valid at* $\mathcal{X}, x$ (notation: $\mathcal{X}, x \Vdash A$) if A is intuitionistically true at world x under all intuitionistic valuations;
- *intuitionistically valid in* $\mathcal{X}$ (notation: $\mathcal{X} \Vdash A$) if A is intuitionistically valid at all worlds of $\mathcal{X}$.

The set of all intuitionistic formulas valid in $\mathcal{X}$ is denoted by $\mathbf{IL}(\mathcal{X})$ (the *intermediate logic of* $\mathcal{X}$)

Let us recall sufficient conditions for validity of AG and Br_2.

LEMMA 1. *For a Kripke* **S4**-*frame* $F = (W, R)$

(1) $F \models AG$ *iff* F *is Nöthereian, i.e. it does not contain infinite ascending chains:* $x_0 R x_1 R \ldots$

(2) If R *is a partial order, then* $F \Vdash Br_2$ *if every world has at most two immediate successors:* $\forall x \exists y, z\ (xRy\ \&\ xRz\ \&\ R(x) = \{x\} \cup R(y) \cup R(z))$.

Let us also recall two standard facts:

LEMMA 2. *Let* θ, ψ *be a modal and an intuitionistic valuation in the same topological space such that for any* $s \in PV$

$$\psi(s) = \Box\theta(s).$$

Then for any intuitionistic A

$$\psi^I(A) = \theta(A^T).$$

COROLLARY 3. *Let* $\mathcal{X}$ *be a topological space,* A *an intuitionistic formula. Then*

(1) for any $x \in \mathcal{X}$ $\quad \mathcal{X}, x \Vdash A$ *iff* $\mathcal{X}, x \models A^T$,

(2) $\mathcal{X} \Vdash A$ *iff* $\mathcal{X} \models A^T$.

A **K4**-frame $\mathcal{X} = (X, \Box)$ is associated with a topological space $\mathcal{X}^+ = (X, \Box^+)$ such that $\Box^+ Y = \Box Y \cap Y$. The topological terminology referring to $\mathcal{X}^+$ will be also used for $\mathcal{X}$; so for example, we say that a subset Y is *closed in* $\mathcal{X}$ if it is closed in $\mathcal{X}^+$ (which is equivalent to $\Diamond Y \subseteq Y$); Y is *open in* $\mathcal{X}$ iff $Y \subseteq \Box Y$. So a closed point x may be of two kinds: *reflexive*, with $\Diamond\{x\} = \{x\}$, and *irreflexive*, with $\Diamond\{x\} = \varnothing$.

If $\mathcal{X}$ is corresponds to a Kripke frame (W, R), then $\Diamond Y = R^{-1}(Y)$, so x is closed iff x is R-minimal, and the above reflexivity notion corresponds to the standard one.

LEMMA 4. *Let* x *be a reflexive closed point in a* **K4**-*frame. Then* $x \in \Box V$ *implies* $x \in V$.

Proof. Suppose $x \in \Box V$, but $x \notin V$. Then $V \subseteq -\{x\}$, and so $\Box V \subseteq \Box(-\{x\})$. Thus $x \in \Box(-\{x\})$, i.e. $x \notin \Diamond\{x\}$ — a contradiction. ■

DEFINITION 5. Let $\mathcal{X} = (X, \Box)$ be a neighbourhood frame, $X_1 \subseteq X$. The *restriction of* $\mathcal{X}$ *to* X_1 (or the *subframe obtained by restriction to* X_1, notation: $\mathcal{X} \upharpoonright X_1$) is $\mathcal{X}_1 = (X_1, \Box_1)$, where $\Box_1 V := \Box V \cap X_1$ for $V \subseteq X_1$. X_1 (and $\mathcal{X}_1$) is called *open* if X_1 is open.

LEMMA 6. *Let* $\mathcal{X}_1$ *be an open subframe of a* **K4**-*frame* $\mathcal{X}$. *Let* ψ *be a valuation in* $\mathcal{X}$, ψ_1 *a valuation in* $\mathcal{X}_1$ *such that* $\psi_1(s) = \psi(s) \cap X_1$ *for any* $s \in PV$. *Then for any modal formula* A, $\psi_1(A) = \psi(A) \cap X_1$.

Proof. Easy, by induction on A. Here is the induction step for $A = \Box B$: suppose $\psi_1(B) = \psi(B) \cap X_1$; then
$\psi_1(A) = \Box_1 \psi_1(B) = \Box_1(\psi(B) \cap X_1) = \Box(\psi(B) \cap X_1) \cap X_1 =$
$\Box\psi(B) \cap \Box X_1 \cap X_1 = \Box\psi(B) \cap X_1 = \psi(A) \cap X_1$. ■

LEMMA 7. *If* $\mathcal{X}_1 \subseteq \mathcal{X}$ *is open, then* $\mathbf{ML}(\mathcal{X}) \subseteq \mathbf{ML}(\mathcal{X}_1)$.

Proof. By Lemma 6, if $(\mathcal{X}_1, \psi_1) \not\vDash A$, then $(\mathcal{X}, \psi) \not\vDash A$, where $\psi(s) = \psi_1(s)$ for any $s \in PV$. ■

DEFINITION 8. For a Kripke frame $F = (W, R)$ and a set $V \subseteq W$ we define the *subframe* $F \upharpoonright V := (V, R \cap (V \times V))$. A *subframe of (transitive)* F *generated by a world* x is $F^x := F \upharpoonright \overline{R}(x)$, where $\overline{R}(x) := R(x) \cup \{x\}$.

A frame F is called *rooted* with the root x if $F = F^x$, i.e. if $W = \overline{R}(x)$.

A *p-morphism* from a Kripke frame $F = (W, R)$ onto a Kripke frame $F' = (W', R')$ is a surjective map $f : W \longrightarrow W'$ such that for any $x \in W$

$$f(R(x)) = R'(f(x)).$$

$f : F \twoheadrightarrow F'$ denotes that f is a p-morphism from F onto F'. The following two lemmas are well-known.

LEMMA 9. (Generation Lemma)
$\mathbf{L}(F) = \bigcap\{\mathbf{L}(F^x) \mid x \in W\}$.

LEMMA 10. (P-morphism Lemma)
$f : F \twoheadrightarrow F'$ *implies* $\mathbf{ML}(F) \subseteq \mathbf{ML}(F')$ *(and* $\mathbf{IL}(F) \subseteq \mathbf{IL}(F')$ *if* F, F' *are* **S4**-*frames). More precisely, if* $f : F \twoheadrightarrow F'$ *and for every* $s \in PV$, $\varphi(s) = f^{-1}(\varphi'(s))$, *then*

$$(F, \varphi), x \vDash A \text{ iff } (F', \varphi'), f(x) \vDash A$$

for any world x *and modal formula* A, *and similarly for the intutuionistic case.*

DEFINITION 11. A formula A (modal or intuitionistic) is a *logical consequence of a logic* $\mathbf{\Lambda}$ *(respectively, modal or intermediate) in neighbourhood semantics* (notation: $\mathbf{\Lambda} \vDash_N A$) if A is valid in all neighbourhood $\mathbf{\Lambda}$-frames.

Similarly, A is a *logical consequence of* $\mathbf{\Lambda}$ *in Kripke semantics* (notation: $\mathbf{\Lambda} \vDash_K A$) if A is valid in all Kripke $\mathbf{\Lambda}$-frames.

One can easily check the following

LEMMA 12.

(1) $C_K(\mathbf{\Lambda}) := \{A \mid \mathbf{\Lambda} \vDash_K A\}$ *is the smallest K-complete logic containing* $\mathbf{\Lambda}$.

(2) $C_N(\mathbf{\Lambda}) := \{A \mid \mathbf{\Lambda} \vDash_N A\}$ *is the smallest N-complete logic containing* $\mathbf{\Lambda}$.

So we have

$$\mathbf{\Lambda} \subseteq C_N(\mathbf{\Lambda}) \subseteq C_K(\mathbf{\Lambda}).$$

DEFINITION 13. A modal or intermediate logic $\mathbf{\Lambda}$ is called *relatively complete* if $C_N(\mathbf{\Lambda}) = C_K(\mathbf{\Lambda})$.

We can also consider *finitary logical consequence.*

DEFINITION 14. $\mathbf{\Lambda} \vDash^0_N A$ if $\mathbf{\Lambda}_1 \vDash_N A$ for some finitely axiomatisable $\mathbf{\Lambda}_1 \subseteq \mathbf{\Lambda}$. The relation $\mathbf{\Lambda} \vDash^0_K A$ is defined analogously.

Let
$C^0_K(\mathbf{\Lambda}) := \{A \mid \mathbf{\Lambda} \vDash^0_K A\}$,
$C^0_N(\mathbf{\Lambda}) := \{A \mid \mathbf{\Lambda} \vDash^0_N A\}$.
The following diagram is clear:

$$\begin{array}{ccc} \mathbf{\Lambda} \subseteq C^0_N(\mathbf{\Lambda}) & \subseteq & C_N(\mathbf{\Lambda}) \\ \cap\!\!| & & \cap\!\!| \\ C^0_K(\mathbf{\Lambda}) & \subseteq & C_K(\mathbf{\Lambda}) \end{array}$$

DEFINITION 15. A logic $\mathbf{\Lambda}$ is called

- *TK-compact* if $C^0_K(\mathbf{\Lambda}) = C_K(\mathbf{\Lambda})$,
- *TN-compact* if $C^0_N(\mathbf{\Lambda}) = C_N(\mathbf{\Lambda})$,
- *TKN-compact* if $C_N(\mathbf{\Lambda}) \subseteq C^0_K(\mathbf{\Lambda})$.

Obviously, every finitely axiomatisable logic is both TK-compact and TN-compact. There is also the following diagram of properties:

$$\begin{array}{ccccc} \text{K-completeness} & \Rightarrow & \text{TK-compactness} & \Rightarrow & \text{TKN-compactness} \\ \Downarrow & & & & \\ \text{N-completeness} & \Rightarrow & \text{TN-compactness} & \Rightarrow & \text{TKN-compactness} \end{array}$$

DEFINITION 16.

A set of modal formulas Γ is called *satisfiable* in a frame F if there exists a model M over F and a world x such that $M, x \vDash A$ for every $A \in \Gamma$.

DEFINITION 17. Let $\mathbf{\Lambda}$ be a modal logic. A set of modal formulas Γ is called $\mathbf{\Lambda}$-*N-satisfiable* (respectively, $\mathbf{\Lambda}$-*K-satisfiable*) if it is satisfiable in some neighbourhood (respectively, Kripke) $\mathbf{\Lambda}$-frame. Γ is called *finitely* $\mathbf{\Lambda}$-*N-satisfiable* if every its finite subset is $\mathbf{\Lambda}$-N-satisfiable; the definition of finite $\mathbf{\Lambda}$-K-satisfiability is analogous.

DEFINITION 18. A modal logic $\mathbf{\Lambda}$ is called

- *N-compact* if every finitely $\mathbf{\Lambda}$-N-satisfiable set is $\mathbf{\Lambda}$-N-satisfiable,
- *strongly neighbourhood (SN-) complete* if it is both N-complete and N-compact.

K-compactness and SK-completeness are defined in a similar way.

An equivalent definition of SN-completeness is the following: every $\mathbf{\Lambda}$-consistent set of formulas is $\mathbf{\Lambda}$-N-satisfiable.

3 Ultrabouqets of topological spaces

The notion of an ultrabouqet exists in several versions, cf. [Shehtman, 1998; 1999]. Let us begin with the case of topological spaces.

DEFINITION 19. Let (X_n, x_n), $n \in \omega$ be sets with designated points. Their *bouqet* $\bigvee\limits_{n\in\omega} (X_n, x_n)$ is obtained from the disjoint union $\bigsqcup\limits_{n\in\omega} X_n$ by identifying all points x_n.

We denote the designated point of $\bigvee\limits_{n\in\omega} (X_n, x_n)$ by x_*.

DEFINITION 20. Let $\mathcal{X}_n = (X_n, \Box_n)$ be topological spaces, and for every n, let x_n be a closed point in $\mathcal{X}_n$. Let $\mathcal{U}$ an ultrafilter in ω. Then we define the *ultrabouqet* $\bigvee\limits_{\mathcal{U}}(\mathcal{X}_n, x_n)$ as the bouqet $\bigvee\limits_{n\in\omega} (X_n, x_n)$ with the topology, in which a subset V is open iff the following conditions hold:

(1) every part $V \cap (X_n - \{x_n\})$ is open;

(2) if $x_* \in V$, then $\{n \mid x_n \in \Box_n(V \cap X_n)\} \in \mathcal{U}$.

A particular case of this construction is when every $\mathcal{X}_n$ corresponds to a Kripke frame $F_n = (W_n, R_n)$ with root x_n and $R^{-1}(x_n) = \{x_n\}$. Then (1) and (2) can be written as follows:

(1) $R_n(V \cap (X_n - \{x_n\}) \subseteq V$;

(2) if $x_* \in V$, then $\{n \mid W_n \subseteq V\} \in \mathcal{U}$.

DEFINITION 21. Let $\mathcal{X}_n$, x_n be the same as in Definition 20, ψ_n a valuation in $\mathcal{X}_n$. Then we define the valuation $\psi = \bigvee_{\mathcal{U}} \psi_n$ in $\bigvee_{\mathcal{U}}(\mathcal{X}_n, x_n)$ as follows:
for any propositional variable s,
$x \in \psi(s)$ iff $x \in \psi_n(s)$ (whenever $x \in X_n - \{x_n\}$),
$x_* \in \psi(s)$ iff $\forall^\infty n\, x_n \in \psi_n(s)$,
where for a predicate $\mathcal{P}$, $\forall^\infty n\, \mathcal{P}(n)$ means[2] $\{n \mid \mathcal{P}(n)\} \in \mathcal{U}$.

LEMMA 22. *Let $\mathcal{X}_n$, x_n, ψ_n be the same as in Definition 21. Then for any modal formula A,*

(1) $x \in \psi(A)$ iff $x \in \psi_n(A)$ (for $x \in X_n - \{x_n\}$),

(2) $x_* \in \psi(A)$ iff $\forall^\infty n\, x_n \in \psi_n(A)$.

Proof. (1) Follows easily by induction on A; note that $(X - \{x_n\})$ is open both in $\mathcal{X}$ and $\mathcal{X}_n$.

(2) Also by induction, cf. [Shehtman, 1998, Lemma 5.5]. ■

LEMMA 23. *Let $\mathcal{X}_n$, x_n be the same as in Definition 20, $(\mathcal{X}, x_*) = \bigvee_{\mathcal{U}}(\mathcal{X}_n, x_n)$. Then for any modal formula A,*

(1) $\mathcal{X}, x \vDash A$ iff $\mathcal{X}_n, x \vDash A$ (whenever $x \in X_n - \{x_n\}$),

(2) $\mathcal{X}, x_* \vDash A$ iff $\forall^\infty n\, \mathcal{X}_n, x_n \vDash A$.

Proof. (1) (Only if.) Assume $\mathcal{X}, x \vDash A$ and consider an arbitrary valuation ψ_n in $\mathcal{X}_n$. Let ψ be "the same" valuation in $\mathcal{X}$, i.e. $\psi(s) = \psi_n(s)$ for every $s \in PV$. By our assumption, $x \in \psi(A)$; hence $x \in \psi_n(A)$ by Lemma 22 (1).

(If.) Assume $\mathcal{X}_n, x \vDash A$ and consider an arbitrary valuation ψ in $\mathcal{X}$. Let ψ_n be its "restriction" to $\mathcal{X}_n$, i.e. $\psi_n(s) = \psi(s) \cap X_n$ for every $s \in PV$[3]. By assumption, $x \in \psi(A)$; hence $x \in \psi_n(A)$ by Lemma 22 (1).

(2) (Only if.) Assume $\mathcal{X}, x_* \vDash A$ and suppose $\forall^\infty n\, \mathcal{X}_n, x_n \vDash A$ does not hold. Since $\mathcal{U}$ is an ultrafilter, this implies $\forall^\infty n\, \mathcal{X}_n, x_n \nvDash A$. Then consider valuations ψ_n in $\mathcal{X}_n$ such that

- $x_n \notin \psi_n(A)$ if $\mathcal{X}_n, x_n \nvDash A$;

[2] $\forall^\infty$ is read as "for almost all".

[3] More precisely, this means: $x \in \psi(s)$ iff $x \in \psi_n(s)$ (for $x \in X - \{x_n\}$) and $x_* \in \psi(s)$ iff $x_n \in \psi_n(s)$.

- ψ_n is arbitrary otherwise.

Let $\psi = \bigvee_{\mathcal{U}} \psi_n$ (Definition 21); then $\forall^\infty n\, x_n \in \psi_n(\neg A)$ implies $x_* \in \psi(\neg A)$ by Lemma 22. This contradicts our assumption.

(2) (If.) Assume $\forall^\infty n\, \mathcal{X}_n, x_n \vDash A$ and consider an arbitrary valuation ψ in $\mathcal{X}$. For each n, take a valuation ψ_n in $\mathcal{X}_n$ such that $\psi_n(s) = \psi(s) \cap X_n$ for any $s \in PV$. Then $\psi = \bigvee_{\mathcal{U}} \psi_n$; in fact, $x_* \in \psi(s)$ iff $\forall n\, x_n \in \psi_n(s)$ iff $\exists n\, x_n \in \psi_n(s)$. So $x_* \in \psi(s)$ implies $\forall^\infty n\, x_n \in \psi_n(s)$, and $x_* \notin \psi(s)$ implies $\neg\forall^\infty n\, x_n \in \psi_n(s)$. By assumption, $\forall^\infty n\, x_n \in \psi_n(A)$; hence $x_* \in \psi(A)$ by Lemma 22 (2). ■

LEMMA 24. *Let $\mathcal{X}_n$, x_n, $\mathcal{X}$ be the same as in Lemma 23. Then for any intuitionistic formula A,*

(1) $\mathcal{X}, x \vDash A$ iff $\mathcal{X}_n, x \vDash A$ (whenever $x \in X_n - \{x_n\}$),

(2) $\mathcal{X}, x_* \vDash A$ iff $\forall^\infty n\, \mathcal{X}_n, x_n \vDash A$.

Proof. Follows readily from Lemma 23 and Corollary 3. ■

4 Ultrabouqets of K4-frames

Now let us extend the notion of an ultrabouqet to neighbourhood **K4**-frames.

DEFINITION 25. Let $\mathcal{X}_n = (X_n, \Box_n)$, $n \in \omega$ be a family of **K**4-frames, with designated closed points x_n which are all reflexive or all irreflexive. Let $(X, x_*) = \bigvee_{n\in\omega} (X_n, x_n)$ be the corresponding bouqet, $\mathcal{U}$ an ultrafilter in ω. For $V \subseteq X$ we put

$$V_n := V \cap X_n{}^4,$$
$$\Box V := V^1 \cup V^0,$$

where

$$V^1 := \bigcup_n (\Box_n V_n - \{x_n\}),$$

and

$$V^0 := \begin{cases} \{x_*\} & \text{if } \forall^\infty n\, x_n \in \Box_n V_n; \\ \varnothing & \textit{otherwise}. \end{cases}$$

The frame $(X, \Box)$ is called the *ultrabouqet* of the family $(\mathcal{X}_n, x_n)_{n\in\omega}$ w.r.t $\mathcal{U}$ and denoted by $\bigvee_{\mathcal{U}} (\mathcal{X}_n, x_n)$.

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[4]More precisely, $y \in V_n$ iff ($y \neq x_n$ & $y \in V \cap X_n$ or $y = x_n$ & $x_* \in V$).

Note that in the reflexive case $x_* \in \Box V$ implies $x_* \in V$ (and thus x_* is reflexive). In fact, if $\forall^\infty n\, x_n \in \Box_n V_n$, then for some n, $x_n \in \Box_n V_n$; hence $x_n \in V_n$ by Lemma 4, i.e. $x_* \in V$.

DEFINITION 26. Let $\mathcal{X}_n$, x_n be the same as in Definition 25, and let ψ_n be valuations in $\mathcal{X}_n$. Consider the valuation $\psi = \bigvee_{\mathcal{U}} \psi_n$ in $\bigvee_{\mathcal{U}}(\mathcal{X}_n, x_n)$ such that for any $s \in PV$,
$x \in \psi(s)$ iff $x \in \psi_n(s)$ (whenever $x \in X_n - \{x_n\}$),
$x_* \in \psi(s)$ iff $\forall^\infty n\, x_n \in \psi_n(s)$.

LEMMA 27. *Let $\mathcal{X}_n$, x_n, ψ_n, ψ be the same as in Definition 26. Then for any modal formula A,*

(1) $x \in \psi(A)$ iff $x \in \psi_n(A)$ (for $x \in X_n - \{x_n\}$),

(2) $x_* \in \psi(A)$ iff $\forall^\infty n\, x_n \in \psi_n(A)$.

Proof. Both statements are proved by induction on the length of A. Let us consider the only nontrivial case: $A = \Box B$.

(1) We have:

$$x \in \psi(A) = \Box\psi(B) \text{ iff } x \in \psi(B)^1 \text{ iff } x \in \Box_n \psi(B)_n,$$

$$x \in \psi_n(A) \text{ iff } x \in \Box_n \psi_n(B).$$

Since x_n is closed, we also have $x \in \Box_n(X_n - \{x_n\})$, and thus

$$x \in \Box_n \psi(B)_n \text{ iff } x \in \Box_n(\psi(B)_n - \{x_n\}),$$

$$x \in \Box_n \psi_n(B) \text{ iff } x \in \Box_n(\psi_n(B) - \{x_n\}).$$

By induction hypothesis,

$$\psi(B)_n - \{x_n\} = \psi_n(B) - \{x_n\},$$

hence

$$x \in \psi(A) \text{ iff } x \in \psi_n(A).$$

(2) `Reflexive case.` By Definition 25 we have:

$$(\sharp) \quad x_* \in \psi(A) = \Box\psi(B) \text{ iff } \forall^\infty n\, x_n \in \Box_n \psi(B)_n.$$

Now assume $x_* \in \psi(A)$. By the remark after Definition 25 it follows that $x_* \in \psi(B)$, and thus for any n, $x_n \in \psi(B)_n$. Hence by induction hypothesis (1),

(♯♯) $\psi(B)_n = \{x_n\} \cup (\psi(B)_n - \{x_n\}) = \{x_n\} \cup (\psi_n(B) - \{x_n\}) = \psi_n(B) \cup \{x_n\}$.

By induction hypothesis (2), $x_* \in \psi(B)$ implies $\forall^\infty n\ x_n \in \psi_n(B)$, and thus from (♯♯) we have

$$(\sharp\sharp\sharp) \quad \forall^\infty n\ \psi_n(B) = \psi(B)_n.$$

Eventually from (♯♯) and (♯♯♯) we obtain

(♮) $\forall^\infty n\, x_n \in \Box_n \psi_n(B) = \psi_n(A)$.

Conversely, assume (♮). Then by Lemma 4, $\forall^\infty n\ x_n \in \psi_n(B)$, and thus $x_* \in \psi(B)$ by induction hypothesis (2). Hence by the same argument as above we obtain (♯♯♯). Now it follows that $\forall^\infty n\, x_n \in \Box_n \psi(B)_n$, which implies $x_* \in \psi(A)$ by (♯).

`Irreflexive case.`

By Definition 4,

$$x_* \in \psi(A) \text{ iff } \forall^\infty n\, x_n \in \Box_n \psi(B)_n.$$

By induction hypothesis,

$$\psi_n(B) - \{x_n\} = \psi(B)_n - \{x_n\},$$

hence

$$x_n \in \Box_n(\psi_n(B) - \{x_n\}) \text{ iff } x_n \in \Box_n(\psi(B)_n - \{x_n\}).$$

Since $\Diamond\{x_n\} = \varnothing$, we also have $x_n \in \Box_n(-\{x_n\})$, and thus

$$x_n \in \Box_n \psi_n(B) \text{ iff } x_n \in \Box_n \psi(B)_n.$$

This eventually implies

$$x_* \in \psi(A) \text{ iff } \forall^\infty n\, x_n \in \Box_n \psi_n(B) = \psi_n(A).$$

■

Hence similarly to Lemma 23, we obtain

LEMMA 28. *Let $\mathcal{X}_n$, x_n be the same as in Definition 25, $(\mathcal{X}, x_*) = \bigvee_{\mathcal{U}} (\mathcal{X}_n, x_n)$ Then for any modal formula A,*

(1) $\mathcal{X}, x \vDash A$ iff $\mathcal{X}_n, x \vDash A$ (whenever $x \in X_n - \{x_n\}$),

(2) $\mathcal{X}, x_* \vDash A$ iff $\forall^\infty n\, \mathcal{X}_n, x_n \vDash A$.

In particular, it follows that an ultrabouqet of **K4**-frames is a **K4**-frame.

5 TKN-noncompactness above Grz

DEFINITION 29. Let us define modal formulas β_n, γ_n by induction.

$$\begin{array}{ll} \beta_0 = \Box p, & \gamma_0 = \Box\neg p, \\ \beta_1 = \neg p \wedge \Diamond\beta_0 \wedge \neg\Diamond\gamma_0, & \gamma_1 = p \wedge \Diamond\gamma_0 \wedge \neg\Diamond\beta_0, \\ \beta_{n+1} = \Diamond\beta_n \wedge \Diamond\gamma_{n-1} \wedge \neg\Diamond\gamma_n, & \gamma_{n+1} = \Diamond\gamma_n \wedge \Diamond\beta_{n-1} \wedge \neg\Diamond\beta_n. \end{array}$$

Also let

$$\alpha_n = \Diamond\beta_{n+1} \wedge \Diamond\gamma_{n+1} \wedge \neg\Diamond\beta_{n+2} \wedge \neg\Diamond\gamma_{n+2},$$

$$\varepsilon_n = \Diamond\alpha_n \wedge \Diamond\beta_{n+2},\ \theta_n = \varepsilon_{n+1} \wedge \neg\Diamond\alpha_n,$$

$\delta_n = \varepsilon_n \to \Diamond\theta_n$, $\mathbf{\Lambda}_1 = \mathbf{Grz}+\{\delta_n \mid n \geq 0\}$, $\mathbf{\Lambda}_1^{(n)} = \mathbf{Grz}+\{\delta_m \mid n \geq m \geq 0\}$.

LEMMA 30. *The following formulas are* **S4***-theorems:*

(1) $\beta_n \to \Diamond\beta_m$ *for* $n \geq m \geq 0$*,*

(2) $\beta_n \to \Diamond\gamma_0$ *for* $n \geq 2$*,*

(3) $\varepsilon_n \to \Diamond\beta_1$ *for* $n \geq 0$*.*

Proof. By definition, $\mathbf{S4} \vdash \beta_n \to \Diamond\beta_{n-1}$. Hence by induction it follows that $\mathbf{S4} \vdash \beta_n \to \Diamond\beta_m$ for $n \geq m$. Since $\mathbf{S4} \vdash \beta_2 \to \Diamond\gamma_1,\ \gamma_1 \to \Diamond\gamma_0$ by definition, it follows that $\mathbf{S4} \vdash \beta_2 \to \Diamond\gamma_0$, and thus we obtain $\mathbf{S4} \vdash \beta_n \to \Diamond\gamma_0$ for $n \geq 2$.

For the proof of (3), note that $\mathbf{S4} \vdash \varepsilon_n \to \Diamond\alpha_n,\ \alpha_n \to \Diamond\beta_{n+1}$ by definition and $\mathbf{S4} \vdash \beta_{n+1} \to \Diamond\beta_1$ by (1). ■

LEMMA 31. *A topological space $\mathcal{X}$ refutes AG iff there exist sets X_0, $X_1 \subseteq X$ such that*

$$X_0 \neq \varnothing,\ X_0 \cap X_1 = \varnothing,\ X_0 \subseteq \Diamond X_1,\ X_1 \subseteq \Diamond X_0.$$

Proof. (If.) Take a valuation φ such that $\varphi(p) = X_0$ and consider the model $(\mathcal{X}, \varphi)$. Then we have: $X_1 \subseteq \varphi(\neg p)$, $X_1 \subseteq \varphi(\Diamond p)$, and thus

$$X_0 \subseteq \varphi(\Diamond(\neg p \wedge \Diamond p)).$$

Hence $(\mathcal{X}, \varphi) \vDash p \to \Diamond(\Diamond p \wedge \neg p)$, and therefore

$$X_0 \subseteq \varphi(p \wedge \Box(p \to \Diamond(\neg p \wedge \Diamond p))).$$

Since $X_0 \neq \varnothing$, this implies that $\mathcal{X}$ refutes AG.

(Only if.) Consider a model $(\mathcal{X}, \varphi)$ and a point u such that

$$u \in \varphi(p \wedge \Box(p \to \Diamond(\neg p \wedge \Diamond p))).$$

Take a neighbourhood V of x such that $V \subseteq \varphi(p \to \Diamond(\neg p \wedge \Diamond p))$ and put $X_0 := \varphi(p) \cap V$, $X_1 := \varphi(\neg p \wedge \Diamond p) \cap V$. It follows that X_0, X_1 are the sets required. ■

Remark. This lemma means that AG behaves like a subframe formula for topological spaces: $\mathcal{X} \not\models AG$ iff there exists a subreduction from $\mathcal{X}$ onto a two-element cluster, i.e. an interior map from a subspace of $\mathcal{X}$ onto the two-element space with the weakest topology.

LEMMA 32. *A topological space $\mathcal{X}$ refutes AG iff there exist sets $Y_n \subseteq X$, $n \geq 0$ such that $Y_0 \neq \varnothing$ and for any n,*

$$Y_n \cap Y_{n+1} = \varnothing,\ Y_n \subseteq \Diamond Y_{n+1}.$$

Proof. (If.) Let $Y = \bigcup_{n\in\omega} Y_n$. For $y \in Y$ put

$$m(y) := \min\{n \mid y \in Y_n\},$$

and let

$$X_0 := \{y \in Y \mid m(y) \text{ is even}\},$$
$$X_1 := \{y \in Y \mid m(y) \text{ is odd}\}.$$

Let us show that X_0, X_1 satisfy the conditions from the previous lemma. In fact, $X_0 \cap X_1 = \varnothing$ is trivial, and obviously, $X_0 \supseteq Y_0 \neq \varnothing$. It remains to prove by induction that

$$(*) \quad \forall n \forall y \in Y\ (m(y) = n \Rightarrow y \in \Diamond X_0 \cap \Diamond X_1),$$

i.e. that X_0, X_1 are dense in Y. In fact, assume that (*) holds for any $k < n$. Let $m(y) = n$, then $y \in Y_n \subseteq \Diamond Y_{n+1}$. If n is even, then obviously, $y \in X_0 \subseteq \Diamond X_0$; so we have to show that $y \in \Diamond X_1$.

Take an arbitrary neighbourhood V of y; then it contains a point $z \in Y_{n+1}$. Since $Y_n \cap Y_{n+1} = \varnothing$, we have either $m(z) = n+1$ (in which case $z \in X_1$, by definition), or $m(z) < n$. In the latter case $z \in \Diamond X_1$ by induction hypothesis, and thus $V \cap X_1 \neq \varnothing$. Hence $y \in \Diamond X_1$.

If n is odd, the argument is similar.

(Only if.) Take the sets X_0, X_1 from Lemma 31 and put

$$Y_n := \begin{cases} X_0 & \text{if } n \text{ is even;} \\ X_1 & \text{if } n \text{ is odd.} \end{cases}$$

■

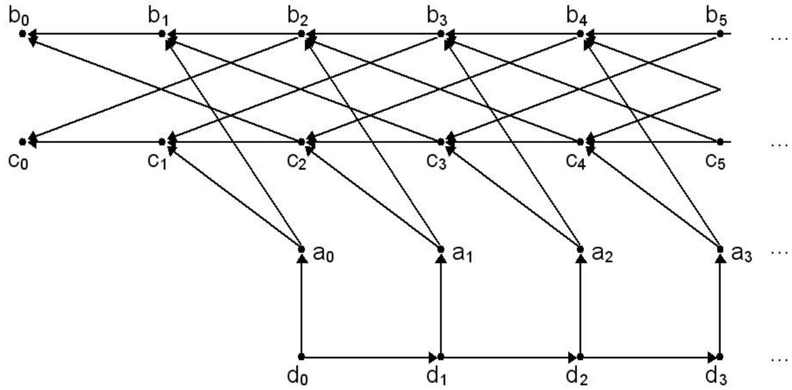


Figure 1.

LEMMA 33. $\neg\varepsilon_0 \in C_N(\mathbf{\Lambda}_1)$.

Proof. For a topological space $\mathcal{X}$, suppose $\mathcal{X} \vDash \mathbf{\Lambda}_1$, but $\varphi(\varepsilon_0) \neq \varnothing$ for some valuation φ in $\mathcal{X}$. Let $Y_n := \varphi(\theta_n)$. Then by definition, $Y_n \subseteq \varphi(\varepsilon_{n+1})$, and $\varphi(\varepsilon_{n+1}) \subseteq \Diamond Y_{n+1}$, since $\mathcal{X} \vDash \delta_{n+1}$. Hence $Y_n \subseteq \Diamond Y_{n+1}$.

On the other hand, $Y_n \cap Y_{n+1} = \varnothing$, since

$$Y_n \subseteq \varphi(\varepsilon_{n+1}) \subseteq \varphi(\Diamond\alpha_{n+1}),\ Y_{n+1} \subseteq \varphi(\neg\Diamond\alpha_{n+1}).$$

Finally, $\varnothing \neq \varphi(\varepsilon_0) \subseteq \Diamond Y_0$, since $\mathcal{X} \vDash \delta_0$, and thus $Y_0 \neq \varnothing$.

So by Lemma 32, $\mathcal{X} \nvDash AG$, which contradicts our assumption. ■

DEFINITION 34. $\Phi = (W, \leq)$ (Fine's frame) is the partially order set shown in Fig. 1.

DEFINITION 35. Let $\Phi_n = (W_n, \leq)$ be the restriction of Φ to the set $W_n := W - \{d_m \mid m \geq n+2\}$ (the n-truncated Fine's frame)[5].

LEMMA 36. $\Phi_n \vDash \delta_m$ *for any* $m \leq n$.

Proof. For an arbitrary model (Φ_n, φ), we have to prove that $\varphi(\varepsilon_m) \subseteq \varphi(\Diamond\theta_m)$. So let us assume $x \vDash \varepsilon_m$ and show that $x \vDash \Diamond\theta_m$ (in this model). First note that either (1) or (2) holds:

[5]To simplify notation, we use the same symbol $\leq$ for the relation in Φ_n.

(1) $b_0 \vDash p,\ c_0 \vDash \neg p$,

(2) $c_0 \vDash p,\ b_0 \vDash \neg p$.

In fact, if $b_0, c_0 \vDash p$, then $\varphi(\Box\neg p) = \varnothing$, since either b_0 or c_0 is accessible from any world of Φ_n. Thus $\varphi(\gamma_0) = \varnothing$, which implies $\varphi(\beta_k) = \varnothing$ for any $k \geq 2$) (remember that $\mathbf{S4} \vdash \beta_k \to \Diamond\gamma_0$ by Lemma 30). This contradicts $x \vDash \varepsilon_m$.

A similar argument shows, that p cannot be false at both $b_0,\ c_0$.

Next, if (1) holds, by induction we obtain that for any k

(3) $\quad \varphi(\beta_k) = \{b_k\},\ \varphi(\gamma_k) = \{c_k\}$.

In fact, we obviously have

$$b_0 \in \varphi(\beta_0),\ c_0 \in \varphi(\gamma_0)$$

and thus $\varphi(\neg\Diamond\gamma_0) \subseteq \{b_0, b_1\}$. Since $b_0 \vDash p$, it follows that $\varphi(\beta_1) \subseteq \{b_1\}$.

On the other hand, $\Phi_n \vDash \varepsilon_m \to \Diamond\beta_1$ by Lemma 30 and soundness; thus $x \vDash \Diamond\beta_1$, and so $\varphi(\beta_1) \neq \varnothing$, and the only remaining option is $\varphi(\beta_1) = \{b_1\}$.

A similar argument shows that $\varphi(\gamma_1) = \{c_1\}$.

Now we can apply induction for the proof of (3); for the induction step note that for any y

$$y \vDash \beta_{k+1} \text{ iff } y \leq b_k \ \&\ y \leq c_{k-1} \ \&\ y \not\leq c_k \text{ iff } y = b_{k+1},$$

and similarly for $y \vDash \gamma_{k+1}$.

Next, if (2) holds, in the same way we obtain

(4) $\quad \varphi(\beta_k) = \{c_k\},\ \varphi(\gamma_k) = \{b_k\}$.

Now since

$$y = a_k \text{ iff } y \leq b_{k+1} \ \&\ y \leq c_{k+1} \ \&\ y \not\leq b_{k+2} \ \&\ y \not\leq c_{k+2},$$

it follows that (in any case)

(5) $\quad \varphi(\alpha_k) = \{a_k\}$.

By assumption, $x \vDash \varepsilon_m$, so we have

$$x \leq a_m \text{ and } (x \leq b_{m+2} \text{ or } x \leq c_{m+2}).$$

Therefore $x \leq d_m$.

But $m \leq n$, and thus $d_{m+1} \in \Phi_n$. It remains to note that $d_{m+1} \vDash \theta_m$. In fact, we can again apply (3), (4), (5): $d_{m+1} \vDash \varepsilon_{m+1}$, since $d_{m+1} \leq a_{m+1}, b_{m+3}, c_{m+3}$; at the same time $d_{m+1} \not\vDash \Diamond\alpha_m$, since $d_{m+1} \not\leq a_m$.

So we obtain $x \vDash \Diamond\theta_m$, as required. ∎

LEMMA 37. $\Phi_n \vDash \mathbf{\Lambda}_1^{(n)}$

Proof. Since Φ_n is Nötherian, by Lemma 1 it follows that $\Phi_n \vDash AG$. The remaining axioms of $\mathbf{\Lambda}_1^{(n)}$ are valid, by Lemma 36. ■

LEMMA 38. $\Phi_n \nvDash \neg\varepsilon_0$

Proof. Take a valuation φ in Φ_n such that $\varphi(p) = \{b_0,\ c_1\}$. The same induction as in the proof of Lemma 36 shows that

$$\varphi(\beta_k) = \{b_k\},\ \varphi(\gamma_k) = \{c_k\}.$$

This implies $a_0 \in \varphi(\alpha_0)$, and thus $d_0 \in \varphi(\varepsilon_0)$. ■

THEOREM 39. *The logic $\mathbf{\Lambda}_1$ is TKN-noncompact.*

Proof. $\neg\varepsilon_0 \in C_N(\mathbf{\Lambda}_1)$, by Lemma 33.

On the other hand, $\neg\varepsilon_0 \notin C_K^0(\mathbf{\Lambda}_1)$. In fact, suppose $\mathbf{S4} + A \vDash_K \neg\varepsilon_0$ for some $A \in \mathbf{\Lambda}_1$. Then A is provable in **Grz** with a finite set of extra axioms, i.e. $A \in \mathbf{\Lambda}_1^{(n)}$ for some finite n. It follows that $\mathbf{\Lambda}_1^{(n)} \vDash_K \neg\varepsilon_0$, which contradicts Lemmas 37 and 38. ■

6 Relative incompleteness above Grz

Now let us slightly modify the counterexample from the previous Section to obtain another counterexample. We use the same special formulas as in Definition 29.

DEFINITION 40. Let

$$\delta'_n = \varepsilon_0 \to \Box\delta_n,\ \mathbf{\Lambda}_2 = \mathbf{Grz} + \{\delta'_n \mid n \geq 0\}.$$

LEMMA 41.

(1) $\Phi_n \vDash \delta'_m$ *for any* $m \leq n$.

(2) $\Phi_n, x \vDash \delta'_m$ *for any* $x \neq d_0$ *and for any* m.

Proof. Consider an arbitrary model $M = (\Phi_n, \varphi)$, and assume $M, x \vDash \varepsilon_0$. The proof of Lemma 36 shows that for any k, we have either

(3) $\varphi(\beta_k) = \{b_k\},\ \varphi(\gamma_k) = \{c_k\}$

or

(4) $\varphi(\beta_k) = \{c_k\},\ \varphi(\gamma_k) = \{b_k\}$,

and also

(5) $\varphi(\alpha_k) = \{a_k\}$.

Thus $x \leq a_0$ and either $x \leq b_2$ or $x \leq c_2$, which is possible only if $x = d_0$. So (2) follows readily.

The claim (1) is a trivial consequence of Lemma 36: $\Phi_n \vDash \delta_m$ for any $m \leq n$. ■

LEMMA 42. $\neg\varepsilon_0 \in C_K(\mathbf{\Lambda}_2)$.

Proof. Suppose there is a Kripke frame $F = (V, R)$ such that $F \vDash \mathbf{\Lambda}_2$, but $x_0 \vDash \varepsilon_0$ for some $x_0 \in V$ in some model over F. Then there exists an infinite ascending chain starting from x_0 such that $x_n R x_{n+1}$ and $x_n \vDash \varepsilon_n$. In fact, if $x_n \vDash \varepsilon_n$, we also have $x_n \vDash \delta_n$ $(= \varepsilon_n \to \Diamond\theta_n)$, and thus there exists $x_{n+1} \in R(x_n)$ such that $x_{n+1} \vDash \theta_n$ (and so $x_{n+1} \vDash \varepsilon_{n+1}$).

But then F is not Nötherian, which contradicts $F \vDash \mathbf{Grz}$. ■

LEMMA 43. *Let $\mathcal{U}$ be a non-principal ultrafilter on ω, $\mathcal{X} = \bigvee_{\mathcal{U}}(\Phi_n, d_0)$. Then $\mathcal{X} \vDash \mathbf{\Lambda}_2$, but $\mathcal{X} \nvDash \neg\varepsilon_0$.*

Proof. Let x_0 be the root of $\mathcal{X}$. By Lemmas 23 and 41, we have: $\mathcal{X}, x \vDash \delta'_m$ for any $x \neq x_0$, for any m. We also have $\mathcal{X}, x_0 \vDash \delta'_m$ since $\{n \mid n \geq m\} \subseteq \{n \mid \Phi_n, d_0 \vDash \delta'_m\}$ and $\mathcal{U}$ is non-principal.

Thus $\mathcal{X} \vDash \delta'_m$.

As we know, every Φ_n validates **Grz** (Lemma 37), so $\mathcal{X} \vDash \mathbf{Grz}$, by Lemma 23.

On the other hand, ε_0 is satisfiable at Φ_n, d_0 for any n, so it is satisfiable at $\mathcal{X}, x_0$, by Lemma 22. Hence $\mathcal{X} \nvDash \neg\varepsilon_0$. ■

THEOREM 44. *$\mathbf{\Lambda}_2$ is relatively incomplete.*

Proof. In fact, $\neg\varepsilon_0 \in (C_K(\mathbf{\Lambda}_2) - C_N(\mathbf{\Lambda}_2))$ by Lemmas 42, 43. ■

We also have

THEOREM 45. *$\mathbf{\Lambda}_2$ is TK-noncompact.*

Proof. Almost the same as for Theorem 39. We already know that $\neg\varepsilon_0 \in C_K(\mathbf{\Lambda}_2)$. To show that $\neg\varepsilon_0 \notin C^0_K(\mathbf{\Lambda}_2)$, it suffices to note that $\mathbf{\Lambda}_2^{(n)} \nvDash_K \neg\varepsilon_0$, where

$$\mathbf{\Lambda}_2^{(n)} = \mathbf{Grz} + \{\delta'_m \mid 0 \leq m \leq n\}.$$

This follows from Lemmas 41, 38. ■

7 TK-noncompactness for intermediate logics

This Section is an intuitionistic analogue of Section 5. Let us first define some intuitionistic formulas.

DEFINITION 46.

$$\begin{array}{ll} B'_0 = \neg(p \wedge q), & C'_0 = \neg(\neg p \wedge q), \\ B'_1 = C'_0 \to B'_0 \vee q, & C'_1 = B'_0 \to C'_0 \vee p, \\ B'_{n+1} = C'_n \to B'_n \vee C'_{n-1}, & C'_{n+1} = B'_n \to C'_n \vee B'_{n-1} \end{array}$$

for $n \geq 0$.

Also let

$$A'_n = B'_{n+2} \wedge C'_{n+2} \to B'_{n+1} \vee C'_{n+1},$$

$$E'_n = A'_n \vee B'_{n+2},\ D'_n = (A'_n \to E'_{n+1}) \to E'_n,$$

$$\mathbf{\Lambda}_3 = \mathbf{H} + \{D'_n \mid n \geq 0\} + Br_2,$$

$$\mathbf{\Lambda}_3^{(n)} = \mathbf{H} + \{D'_m \mid n \geq m \geq 0\} + Br_2.$$

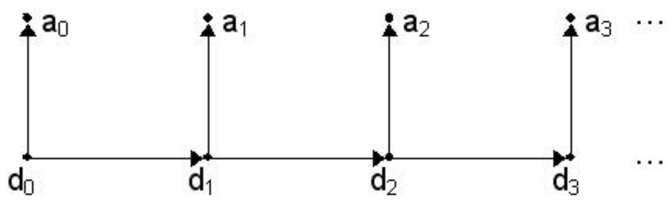
LEMMA 47. *If $m \leq n$, then*

$$\mathbf{H} \vdash B'_m \to B'_n,\ C'_m \to C'_n,\ B'_m \to C'_{n+2},\ C'_m \to B'_{n+2}.$$

Proof. By induction; note that $\mathbf{H} \vdash B'_m \to B'_{m+1},\ C'_m \to C'_{m+1}, B'_m \to C'_{m+2}, C'_m \to B'_{m+2}$, by definition. ■

DEFINITION 48. The frame $\Phi^- := \Phi \restriction W^-$, where Φ is Fine's frame (Definition 34), $W^- := \{a_n \mid n \geq 0\} \cup \{d_n \mid n \geq 0\}$ is called the *willow.*

The willow is a tree shown in the picture.



LEMMA 49. $E'_0 \in C_K(\mathbf{\Lambda}_3)$.

Proof. Analogous to the proof of Lemma 33. For a Kripke frame $F = (V, R)$ suppose $F \vDash \mathbf{\Lambda}_3$, while $\varphi(E'_0) \neq V$ for some model (F, φ). Let $e_0 \notin \varphi(E'_0)$. Let us construct a p-morphism from a subframe of F^{e_0} onto the willow Φ^-. This will give us a refutation of Br_2 in F.

Let

$$\mathcal{D}_n := \varphi(\bigwedge_{m<n} A'_m, E'_n),$$

$$\mathcal{A}_n := \varphi(\bigwedge_{m<n} A'_m \wedge B'_{n+2} \wedge C'_{n+2}, B'_{n+1} \wedge C'_{n+1}).$$

Next, for $x \in V$ let

$$N(x) := \{n \mid x \in R^{-1}(\mathcal{A}_n)\},$$

and also

$$\mathcal{A}_n^+ := \{x \in V \mid N(x) = \{n\}\}.$$

Now let us prove some auxiliary facts.

(1) $\mathcal{D}_n \subseteq R^{-1}(\mathcal{D}_{n+1})$.

In fact, for $x \in \mathcal{D}_n$ we have $x \not\Vdash E'_n$, and thus $x \not\Vdash A'_n \to E'_{n+1}$, since $F \Vdash D'_n$, by assumption. So for some $y \in R(x)$

$$y \Vdash A'_n \ \& \ y \not\Vdash E'_{n+1}.$$

Since $x \in \mathcal{A}_n$, we have $y \Vdash \bigwedge\limits_{m<n} A'_m$, and so $y \in \varphi(\bigwedge\limits_{m<n+1} A'_n, E'_{n+1}) = \mathcal{D}_{n+1}$.

(2) $k \leq n \Rightarrow \mathcal{D}_k \subseteq R^{-1}(\mathcal{D}_n)$.

This follows easily from (1) by induction.

(3) $\mathcal{D}_n \subseteq R^{-1}(\mathcal{A}_n)$.

In fact, $x \in \mathcal{D}_n$ implies $x \not\Vdash E'_n$, and thus $x \not\Vdash A'_n$. But then for some $y \in R(x)$

$$y \Vdash B'_{n+2} \wedge C'_{n+2}, \ y \not\Vdash B'_{n+1} \vee C'_{n+1}.$$

Since $x \Vdash \bigwedge\limits_{m<n} A'_m$, it follows that $y \in \mathcal{A}_n$.

(4) $k \leq n \Rightarrow \mathcal{D}_k \subseteq R^{-1}(\mathcal{A}_n)$.

This follows from (2) and (3).

(5) If $x \in R^{-1}(\mathcal{A}_m) \cap R^{-1}(\mathcal{A}_k),\ m > k,$ then $x \in \mathcal{D}_n$ for some $n \leq k$.

In fact, assume $x \in R^{-1}(\mathcal{A}_m) \cap R^{-1}(\mathcal{A}_k)$. Since $y \not\Vdash A'_k$ for any $y \in \mathcal{A}_k$, we also have $x \not\Vdash A'_k$, $x \not\Vdash A'_m$, and so the set $S = \{l \mid x \not\Vdash A'_l\}$ is non-empty.

Let $n = \min \mathrm{S}$. Then obviously, $x \Vdash \bigwedge\limits_{i<n} A'_i$. On the other hand, $x \in R^{-1}(\mathcal{A}_m)$ implies $x \not\Vdash B'_{m+1}$, and thus $x \not\Vdash B'_{n+2}$ (since $n+2 \leq k+2 \leq m+1$ implies $\mathbf{H} \vdash B'_{n+2} \to B'_{m+1}$, by Lemma 47).

Since $n \in S$, we also have $x \not\Vdash A'_n$, and thus $x \in \mathcal{D}_n$.

(6) If $k \leq n,$ then $\mathcal{D}_k \cap \mathcal{A}_n = \varnothing$.

In fact, assume $k \leq n$. Then $\mathcal{D}_k \subseteq R^{-1}(\mathcal{D}_{n+1})$, by (2). By definition,

$$\mathcal{D}_{n+1} \subseteq -\varphi(E'_{n+1}) \subseteq -\varphi(B'_{n+3}),$$

hence

$$\mathcal{D}_k \subseteq R^{-1}(\mathcal{D}_{n+1}) \subseteq -\varphi(B'_{n+3}).$$

On the other hand,

$$\mathcal{A}_n \subseteq \varphi(B'_{n+2}) \subseteq \varphi(B'_{n+3}),$$

by definition and since $\mathbf{H} \vdash B'_{n+2} \to B'_{n+3}$ (Lemma 47). This yields (6).

(7) $\mathcal{A}_n \subseteq \mathcal{A}_n^+$.

In fact, if $x \in \mathcal{A}_n$, then $n \in N(x)$, and thus by (5), either $N(x) = \{n\}$ or $x \in \mathcal{D}_k$ for some $k \leq n$. The latter contradicts (6).

(8) $R^{-1}(\mathcal{A}_n) = R^{-1}(\mathcal{A}_n^+)$.

This follows from the inclusions

$$\mathcal{A}_n \subseteq \mathcal{A}_n^+ \subseteq R^{-1}(\mathcal{A}_n).$$

(9) $R^{-1}(\mathcal{A}_n) \cap \mathcal{A}_m^+ = \varnothing$ if $m \neq n$.

This is obvious by definition.

(10) $\mathcal{D}_m \cap \mathcal{A}_n^+ = \varnothing$ for any m, n.

In fact, $\mathcal{D}_m \subseteq R^{-1}(\mathcal{A}_m)$ by (3), and also

$$\mathcal{D}_m \subseteq R^{-1}(\mathcal{D}_{m+1}) \subseteq R^{-1}(\mathcal{A}_{m+1})$$

by (1), (3). Now (10) follows from (9), since either $n \neq m$ or $n \neq m+1$.

(11) $\mathcal{D}_n \cap \mathcal{D}_m = \varnothing$ for $m \neq n$.

In fact, we may assume $m < n$. By definition, we have $\mathcal{D}_n \subseteq \varphi(A_m)$ and $\mathcal{D}_m \cap \varphi(A_m) = \varnothing$, whence (11) follows.

Now due to (5), (9), (10), (11), we obtain the following partition of the set $V_0 := \{x \mid N(x) \neq \varnothing\}$:

$$V_0 = \bigcup_{n \geq 0} \mathcal{A}_n^+ \cup \bigcup_{n \geq 0} \mathcal{D}_n$$

Then let us define a map $f: V_0 \longrightarrow W^-$ by putting

$$f(x) := \begin{cases} a_n & \text{if } x \in \mathcal{A}_n^+; \\ d_n & \text{if } x \in \mathcal{D}_n. \end{cases}$$

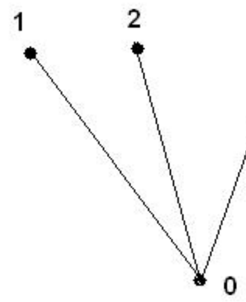
We claim that f is p-morphism from $F_0 := F \restriction V_0$ onto Φ^-.

In fact, if $f(x) = a_n$, then $x \in \mathcal{A}_n^+$, and so obviously, $R(x) \cap \mathcal{A}_m^+ = \varnothing$ for $m \neq n$; and also $R(x) \cap \mathcal{D}_m = \varnothing$ for any m, by (9). This means $f(R(x)) = \{a_n\} = \leq (f(x))$.

If $f(x) = d_n$, then $x \in \mathcal{D}_n$, and so $R(x)$ intersects every $\mathcal{D}_m$ for $m \geq n$, by (2), and thus every $\mathcal{A}_m^+$ for $m \geq n$, by (3) and (6). On the other hand, $x \notin R^{-1}(\mathcal{A}_m^+) = \varnothing$ for $m < n$.

In fact, $x \Vdash A'_m$, by the definition of $\mathcal{D}_m$, while $\mathcal{A}_m^+ \subseteq R^{-1}(\mathcal{A}_m) \subseteq -\varphi(A'_m)$, also by definitions. Thus $f(R(x)) = \leq (f(x))$, i.e. $f : F_0 \twoheadrightarrow \Phi^-$.

Since the willow itself is p-morphically mapped onto the tree $T_{3,2}$ (see the picture),



we have a combined p-morphism $g : F_0 \twoheadrightarrow T_{3,2}$, and from Lemma 10 it follows that Br_2 is refuted in F_0 under the valuation θ such that

$$\theta(p) = g^{-1}(1),\ \theta(q) = g^{-1}(2).$$

But we also obtain a refutation of Br_2 in (V, R) under the valuation θ' such that

$$\theta'(p) = \theta(p) \cup Z \text{ and } \theta'(q) = \theta(q) \cup Z,$$

where $Z := \{x \in V \mid N(x) = \varnothing\}$ (note that θ' is intutuionistic since $R(Z) \subseteq Z$). In fact, $\bigwedge\limits_{0 \leq i \leq 2} (P_i \to \bigvee\limits_{j \neq i} P_j)$ remains true at x, since p, q are true at all points of Z. ■

LEMMA 50. *$\Phi_n \Vdash D'_m$ for any $m \leq n$.*

Proof. Similar to Lemma 36. Assuming that $M = (\Phi_n, \varphi)$ is an intuitionistic Kripke model and $M, x \nVdash E'_m$, let us show that $x \nVdash A'_m \to E'_{m+1}$.

Our first claim is that either (1) or (2) below holds in M:

(1) $b_0 \Vdash p \wedge q,\ c_0 \Vdash \neg p \wedge q$,

(2) $c_0 \Vdash p \wedge q,\ b_0 \Vdash \neg p \wedge q$.

In fact, suppose $b_0 \nVdash p \wedge q,\ c_0 \nVdash p \wedge q$. Since either b_0 or c_0 is accessible from every world of Φ_n, we obtain $x \Vdash \neg(p \wedge q)$, i.e., $x \Vdash B'_0$. But $\mathbf{H} \vdash B'_0 \to B'_{m+2}$, by Lemma 47, hence $x \Vdash B'_{m+2}$, which contradicts $x \nVdash E'_m$. Thus $b_0 \Vdash p \wedge q$ or $c_0 \Vdash p \wedge q$.

In the same way from $\mathbf{H} \vdash C'_0 \to B'_{m+1}$ (Lemma 47) it follows that $b_0 \Vdash \neg p \wedge q$ or $c_0 \Vdash \neg p \wedge q$. Thus only two options, (1) or (2), are possible.

Now assume that (1) holds. Then we have:

(3) $\varphi^\bullet(B'_1) = \{b_1\}$.

Let us first show that $\varphi^\bullet(B'_1) \subseteq \{b_1\}$. In fact, if $y \in \varphi^\bullet(B'_1)$, then $y \Vdash C'_0$, and thus $y \not\leq c_0$ (since $c_0 \nVdash C'_0$), i.e., $y \in \{b_0, b_1\}$. But $b_0 \Vdash q$, thus $b_0 \Vdash B'_1$, and so $y = b_1$.

On the other hand, by our assumption, $x \not\Vdash E'_m$, so we have $x \not\Vdash B'_{m+2}$, and thus $x \not\Vdash B'_1$ (since $\mathbf{H} \vdash B'_1 \to B'_{m+2}$). Therefore $\varphi^\bullet(B'_1) \neq \varnothing$, and (3) follows.

(4) $\varphi^\bullet(B'_0) = \{b_0\}$.

In fact, by (1), $b_0 \Vdash p \wedge q$. On the other hand, $c_0 \not\Vdash p$ by (1) and $b_1 \not\Vdash p$ by (3); thus $y \not\Vdash p$ for any $y \neq b_0$. Hence (4) follows.

(5) $\varphi^\bullet(C'_1) = \{c_1\}$.

The proof is analogous to (3).

(6) $\varphi^\bullet(C'_0) = \{c_0\}$.

The proof is analogous to (4).

(7) for any k, $\varphi^\bullet(B'_k) = \{b_k\}$, $\varphi^\bullet(C'_k) = \{c_k\}$.

This follows by induction from (3)–(6) using the equivalence

$$y = b_{k+1} \text{ iff } y \leq b_k \;\&\; y \leq c_{k-1} \;\&\; y \not\leq c_k.$$

Next, if (2) holds, in the same way we obtain

(8) for any k, $\varphi^\bullet(B'_k) = \{c_k\}$, $\varphi^\bullet(C'_k) = \{b_k\}$.

Now by the same argument as in the proof of Lemma 36, we obtain (in both cases, (1) or (2)):

(9) for any k, $\varphi^\bullet(A'_k) = \{a_k\}$.

By our assumption, $M, x \not\Vdash E'_m$, so $x \leq a_m$, and also $x \leq b_{m+2}$ or $x \leq c_{m+2}$. So it follows that $x \leq d_m$. Next, since $m \leq n$, we have $d_{m+1} \in \Phi_n$ (Definition 35). By (9), (7), (8), $d_{m+1} \in \varphi(A'_m, E'_{m+1})$ — since $d_{m+1} \not\leq a_m$ and $d_{m+1} \leq a_{m+1}$, $d_{m+1} \leq b_{m+3}$, $d_{m+1} \leq c_{m+3}$. Therefore, $x \not\Vdash A'_m \to E'_{m+1}$. ■

LEMMA 51. $\Phi_n \Vdash \mathbf{\Lambda}_3^{(n)}$

Proof. Since Φ_n is of branching 2, by Lemma 1 it follows that $\Phi_n \Vdash Br_2$. The axioms D'_m are valid, by Lemma 50. ■

LEMMA 52. $\Phi_n, d_0 \not\Vdash E'_0$

Proof. Consider a valuation φ on Φ_n such that

$$\varphi(p) = \{b_0\},\ \varphi(q) = \{c_0\}.$$

By the same argument as in the proof of Lemma 50 it follows that for any k

$$\varphi^\bullet(B'_k) = \{b_k\},\ \varphi^\bullet(C'_k) = \{c_k\},\ \varphi^\bullet(A'_k) = \{a_k\}.$$

Since $d_0 \leq a_0$, $d_0 \leq b_2$, we obtain $d_0 \notin \varphi(E'_0)$. ■

THEOREM 53. $\mathbf{\Lambda}_3$ *is TK-noncompact.*

Proof. $E'_0 \in C_K(\mathbf{\Lambda}_3)$ by Lemma 49, and for any n $E'_0 \notin C_K(\mathbf{\Lambda}_3^{(n)})$, by Lemmas 51, 52. Then $E'_0 \notin C^0_K(\mathbf{\Lambda}_3)$, cf. the proof of Theorem 39. ■

8 Relative incompleteness for intermediate logics

This Section is an intuitionistic analogue of Section 5. Now we modify $\mathbf{\Lambda}_3$ to obtain a relatively incomplete logic.

DEFINITION 54. Let

$$D''_n := E'_0 \vee D'_n,\ \ \mathbf{\Lambda}_4 := \mathbf{H} + \{D''_n \mid n \geq 0\}.$$

LEMMA 55.

(1) $\Phi_n \Vdash D''_m$ *for any* $m \leq n$.

(2) $\Phi_n, x \Vdash D''_m$ *for any* $x \neq d_0$ *and for any* m.

Proof. (1) follows readily from Lemma 50.

To prove (2), let us show that $\Phi_n, x \Vdash E'_0$ for any $x \neq d_0$.

In fact, consider a model $M = (\Phi_n, \varphi)$ and suppose $M, x \not\Vdash E'_0$. Then according to the proof of Lemma 50, we obtain that either

$$\varphi^\bullet(B'_k) = \{b_k\},\ \varphi^\bullet(C'_k) = \{c_k\}$$

or

$$\varphi^\bullet(B'_k) = \{c_k\},\ \varphi^\bullet(C'_k) = \{b_k\},$$

and also

$$\varphi^\bullet(A'_k) = \{a_k\}.$$

Hence $x \leq a_0$ and either $x \leq b_2$ or $x \leq c_2$, which eventually implies $x = d_0$. This is a contradiction. ■

LEMMA 56. $E'_0 \in C_K(\mathbf{\Lambda}_4)$.

Proof. Similar to Lemma 49. Suppose $F = (V, R) \Vdash \mathbf{\Lambda}_4$ and $e_0 \notin \varphi(E'_0)$, then $(F, \varphi), e_0 \Vdash D'_n$ for any n. Next, note that the proof of Lemma 49 does not fully use the validity of D'_n; it actually yields that if $\forall n\, (F, \varphi) \Vdash D'_n$ and $\varphi(E'_0) \neq V$, then $F \not\Vdash Br_2$. This implies our assertion. ■

LEMMA 57. *Let* $(\mathcal{X}, x_0) = \bigvee_{\mathcal{U}} (\Phi_n, d_0)$ *be the same as in Lemma 43. Then* $\mathcal{X} \Vdash \mathbf{\Lambda}_4$*, but* $\mathcal{X} \not\Vdash E'_0$.

Proof. By Lemmas 24 and 55, we obtain that every D''_m is valid at any $x \neq x_0$. Since $\Phi_n, d_0 \Vdash D''_m$ for $n \geq m$, from Lemma 24 it also follows that $\mathcal{X}, x_0 \Vdash D''_m$. Since $\Phi_n, d_0 \not\Vdash E'_0$ by Lemma 52, we obtain $\mathcal{X}, x_0 \not\Vdash E'_0$, again by Lemma 24. ■

THEOREM 58. $\mathbf{\Lambda}_4$ *is relatively incomplete.*

Proof. By Lemmas 56, 57, we have $E'_0 \in (C_K(\mathbf{\Lambda}_4) - C_N(\mathbf{\Lambda}_4))$. ■

Remark $\mathbf{\Lambda}_4$ is also TK-noncompact, but $\mathbf{\Lambda}_3$ is slightly simpler.

9 N-compactness for transitive modal logics

DEFINITION 59. A neighbourhood **K4**-frame $\mathcal{X}$ is called *local* T_1 if the corresponding topological space $\mathcal{X}^+$ is local T_1 (in the sense of [Shehtman, 1998]), i.e. if every point is closed in some its neighbourhood.

LEMMA 60. *If $\mathcal{X}$ is a topological space and $\mathcal{X} \vDash AG$, then $\mathcal{X}$ is local T_1.*

Proof. Suppose the contrary, and let $x \in \mathcal{X}$ be a point such that $\Diamond\{x\} \cap U \neq \{x\}$ for any open $U \ni x$, i.e.

$$(\Diamond\{x\} - \{x\}) \cap U \neq \varnothing.$$

Hence
($\sharp$) $x \in \Diamond(\Diamond\{x\} - \{x\})$.

Now Lemma 42 show that $\mathcal{X} \nvDash AG$. In fact, take $X_0 = \{x\}$, $X_1 = (\Diamond\{x\} - \{x\})$; then $X_0 \subseteq \Diamond X_1$ by ($\sharp$), and obviously, $X_0 \neq \varnothing$, $X_0 \cap X_1 = \varnothing$, $X_1 \subseteq X_0$. ■

LEMMA 61.
Every **GL**-*frame is local T_1.*

Proof. By Lemma 60, since $\mathcal{X} \vDash AL$ implies $\mathcal{X}^+ \vDash AG$. The latter is rather well-known: the modality $\Box^+ A = \Box A \wedge A$ satisfies Grzecorczyk axiom if $\Box$ satisfies Löb axiom; the proof is either syntactical or by applying Kripke models. ■

THEOREM 62. *Let $\mathbf{\Lambda} \supseteq$ **K4** be a modal logic, S a set of modal formulas. If S is finitely $\mathbf{\Lambda}$-satisfiable in local T_1-frames, then S is $\mathbf{\Lambda}$-satisfiable.*

Proof. Similar to [Shehtman, 1999, Theorem 3.1]. Suppose $S = \{A_n \mid n \in \omega\}$, $B_n = \bigwedge\limits_{i=0}^{n} A_i$. By assumption, there exists a local T_1 $\mathbf{\Lambda}$-frame $\mathcal{X}_n$, a valuation θ_n and a point x_n such that $(\mathcal{X}_n, \theta_n), x_n \models B_n$.

Let $\mathcal{Y}_n$ be an open subspace of $\mathcal{X}_n$, in which x_n is closed. By Lemma 7, $\mathcal{Y}_n \vDash \mathbf{\Lambda}$, and by Lemma 6, $\mathcal{Y}_n, \psi_n, x_n \vDash B_n$ for some valuation ψ_n. Now there are two cases.

`Case 1.` The set $\{n \mid x_n$ is reflexive in $\mathcal{Y}_n\}$ is infinite.

Let $\{n_1, n_1, \cdots,\}$ be the increasing enumeration of this set; then $n_k \geq k$, and obviously, $\mathcal{Y}_{n_k}, \psi_{n_k}, x_{n_k} \vDash B_k$. To simplify the notation, let $\mathcal{Z}_k = \mathcal{Y}_{n_k}$, $\varphi_k = \psi_{n_k}$, $z_k = x_{n_k}$; thus $z_k \in \varphi_k(B_k)$.

Take a non-principal ultrafilter $\mathcal{U}$ in ω, and consider the ultrabouqet $(\mathcal{Z}, z_*) = \bigvee\limits_{\mathcal{U}} (\mathcal{Z}_n, z_n)$. Then $\mathcal{Z} \models \mathbf{\Lambda}$ by Lemma 28.

On the other hand, $z_k \in \phi_k(B_k)$ implies

$$\forall n \geq k \; z_n \in \varphi_n(A_k),$$

and thus

$$\forall^\infty n \; z_n \in \varphi_n(A_k),$$

since $\mathcal{U}$ is non-principal. Now take the valuation $\varphi = \bigvee\limits_{\mathcal{U}} \varphi_n$. Then by Lemma 27, $z_* \in \varphi(A_k)$, and therefore $(\mathcal{Z}, \varphi), z_* \models S$.

Case 2. The set $\{n \mid x_n$ is reflexive$\}$ is finite. Then the set $\{n \mid x_n$ is irreflexive$\}$ is infinite, and we can repeat the same argument as in Case 1. ■

THEOREM 63.

(1) Every extension of **GL** *is N-compact.*

(2) Every extension of **Grz** *is N-compact.*

Proof. Follows readily from Theorem 62 and Lemmas 61, 60. ■

10 Final remarks

General theory of neighbourhood semantics and other modifications in Kripke semantics in modal logic is far beyond our understanding. However very interesting results on various kinds of semantics were recently obtained by T. Litak [2005], and this gives a hope for further perspectives.

Let us briefly discuss some topics and open problems related to this paper.

10.1 More counterexamples

Our logics $\mathbf{\Lambda}_1$, $\mathbf{\Lambda}_2$ are extensions of **Grz**; it is very likely that similar counterexamples can be constructed above **GL** and between **S4** and **Grz** (cf. [Rybakov, 1977] studying the same properties in Kripke semantics). Moreover, the methods from [Rybakov, 1977; Litak, 2002] allow us to construct a continuum of logics of this kind. However the following question seems more difficult:

Is it true that for any proper extension $\mathbf{\Lambda}$ *of* **S4** *the interval* $[\mathbf{S4}, \mathbf{\Lambda}]$ *contains uncountably many K-incomplete (N-incomplete, etc.) logics?*

Let us also recall another open problem (Kuznetsov, 1974):

Is every intermediate logic N-complete?
and two other related problems:

Is every intermediate logic N-compact?

Is every extension of **S4** *N-compact?*

As for the latter, one can slightly improve Theorem 63, because ultrabouqets can be defined for a larger class of spaces. In fact, let us call a point x in a topological space *weakly closed* if $\Diamond\{x\}$ is a "cluster", i.e. $y \in \Diamond\{x\}$ iff $x \in \Diamond\{y\}$. A space is *weakly local* T_1 if every its point is locally weakly closed. This class of spaces also allows for a certain ultrabouqet construction, and therefore Theorem 63 transfers to extensions of $\mathbf{Grz}_n$, the logic of all Kripke frames with clusters of cardinality $\leq n$. But this argument is not sufficient to cover all logics above **S4**.

10.2 Finitely axiomatisable incomplete logics

In this paper incomplete logics are TK-noncompact, and thus not finitely axiomatisable. But examples of incomplete finitely axiomatisable (f.a.) logics are also known.

N-incomplete f.a. extension of **S4** was first constructed in [Gerson, 1975b], a somewhat simpler N-incomplete f.a. logic above **Grz** can be found in [Shehtman, 1980]; it is obtained as $\mathbf{Grz} + D_0^T$, for D_0 defined below.

A K-incomplete intermediate logic is constructed in [Shehtman, 1977]; see also [Chagrov and Zakharyaschev, 1997, Ch. 6]. The same logic happens to be relatively incomplete, but the proof is quite complicated [Shehtman, 1980]. For the reader's conveneience, let us recall some details of this construction. The basic formulas are almost the same as in Definition 46:

$$\begin{array}{ll} B_0 = q \to p, & C_0 = p \to q, \\ B_1 = C_0 \to B_0 \vee q, & C_1 = B_0 \to C_0 \vee p, \\ B_{n+1} = C_n \to B_n \vee C_{n-1}, & C_{n+1} = B_n \to C_n \vee B_{n-1}. \end{array}$$

Also let

$$A_n = B_{n+2} \wedge C_{n+2} \to B_{n+1} \vee C_{n+1},$$

$$E_n = A_n \vee B_{n+2},\ D_n = (A_n \to E_{n+1}) \to E_n,$$

$$\mathbf{\Lambda}_5 = \mathbf{H} + D_0 + Br_2.$$

Then the logic $\mathbf{\Lambda}_5$ is relatively incomplete; namely,

$$E_0 \in C_K(\mathbf{\Lambda}_5) - C_N(\mathbf{\Lambda}_5).$$

The first part $\mathbf{\Lambda}_5 \vDash_K E_0$ is proved similarly to Lemma 43. To prove $\mathbf{\Lambda}_5 \nvDash_N E_0$, we have to construct a $\mathbf{\Lambda}_5$-space $\mathcal{Y}$ such that $\mathcal{Y} \not\Vdash E_0$.

This construction is nontrivial. The space $\mathcal{Y}$ is obtained from Fine's frame Φ by adding a continuum of extra points, in order to make Br_2 valid. Namely, consider partitions $e = (S_1, S_2, S_3)$ of ω with infinite members (we call them just 'partitions'). A filter $\mathcal{F}$ is called *subordinate* to e if

- $\mathcal{F}$ contains all cofinite subsets of ω,
- $-S_1, -S_2 \notin \mathcal{F}$,
- $-S_3 \in \mathcal{F}$.

One can show that the set of filters subordinate to e is non-empty and satisfies the conditions of Zorn Lemma. So let $\mathcal{F}(e)$ be a maximal element of this set. Let $Y = W \cup \mathcal{E} \cup \mathcal{E}'$, where $\mathcal{E}$ is the set of all partitions, $\mathcal{E}'$ is a copy of $\mathcal{E}$ (more precisely, $\mathcal{E}' = \{e' \mid e \in \mathcal{E}\}$, where $e' = e \times \{\varnothing\}$). Let $\preceq$ be a certain well-ordering of $\mathcal{E}$. Then the space $\mathcal{Y}$ is Y with the topology, where a set V is open iff

- $\leq (V \cap W) \subseteq V$,
- $(\forall e \in \mathcal{E} \cap V)\ \{n \mid a_n \in V\} \in \mathcal{F}(e)$,
- $\forall e, f \in \mathcal{E}\ (e' \in V\ \&\ f \preceq e \Rightarrow f \in V\ \&\ f' \in V)$.

($\leq$ denotes the original relation in Φ).

The question, whether there exists a simpler (say, countable) counterexample of this kind, remains open. Here is another question:

Do there exist f.a. logics that are N-complete, but K-incomplete?

For instance, one can try to axiomatise the ultrabouqets from Sections 4, 6 or the above defined space $\mathcal{Y}$.

10.3 Löwenheim – Skolem property

Classical first order logical consequence does not distinguish between infinite cardinalities: a theory with an infinite model always has a countable model. Unlike this, the relation $\vDash_K$ in modal or intuitionistic logic is quite sensible to cardinality, as the results by S.K. Thomason, A. Chagrov and M. Kracht show, see [Thomason, 1975]; [Chagrov and Zakharyaschev, 1997, Theorem 6.35]; [Kracht, 1999]. What happens in neighbourhood semantics in this respect, is still unclear:

Does there exist an N-complete modal logic that is not determined by any countable neighbourhood frame?

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