# Modal logics of regions and Minkowski spacetime

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**Abstract.** The paper studies modal logics of Kripke frames, in which possible worlds are regions in space with natural accessibility relations. These logics are also interpreted as relativistic temporal logics. Together with an overview, we prove some new results on completeness, decidability, complexity, and finite axiomatizability.

## 1 Introduction

The subject of this paper is between three different themes: temporal logic of relativity, interval temporal logic, and spatial modal logic. Originally these themes were independent, but nowadays there is growing influence between them. A detailed historical overview of all related work might be an interesting, but rather difficult task. So let us give only some introductory remarks and references.

Relativistic temporal logic was first mentioned in Prior's [22] and first axiomatized in [12]. Not so much has been known in this field so far, in comparison to the first-order approach to special and general relativity (for the latter, see e.g. [2] and references therein).

The idea of interval semantics ("a possible world is a time-interval") is traced back to Jean Buridan (14th century), cf. [21]. In a modern setting this idea first appeared in linguistic semantics (e.g. [3]), next in temporal logic [16], [23], [20] and finally in Computer Science logic [15]; see [29], [13] for further references. A simple observation that intervals on the line correspond to points on the half-plane, puts interval logics into the context of two-dimensional modal logics [30], [18].

Basic relations between intervals on the line were identified in [1]: "before", "meets", "overlaps" etc. One can consider modal operators corresponding to these relations, but modal logics involving them all happen to be undecidable [15]. If only some of these modalities are used, a logic may be still decidable. In general, the landscape of interval logics remains yet unclear.

The third kind of logics considered in this paper are modal logics of regions in space. The idea that a region can be a better basic notion in axiomatic geometry than a point, is rather old [8], [31]; an essential work has been done in first-order "pointless" theories of space, cf. [14], [11]. This approach is widely used in Qualitative Spatial Reasoning [7]. Now modal logics are also applied to this field, but mainly in topological setting (cf. [4]; [9], Ch. 16). Quite recently a simple analogy between regions and intervals led to "modal logics of regions" [17], which are discussed below as well.

Informally the main unifying idea of this paper is that points in $(n+1)$-spacetime correspond to regions in $n$-space via cones; so intervals as one-dimensional regions correspond to points in Minkowski 2-space.

The plan of the paper is as follows. Section 2 contains very standard material and notations. Section 3 gives an outline of completeness results in relativistic modal logic. Section 4 shows how these results can be interpreted for logics of balls and intervals, and Section 5 considers modal logics of other regions. In Section 6 we discuss properties of the modality "after" in Minkowski spacetime; exact axiomatizations are still unknown in this case. Section 7 shows that in this area some natural modal logics may be not finitely axiomatizable. In Section 8 we quote some earlier results on complexity and finite model property. Section 9 discusses some results on intuitionistic logic and its extensions, and Section 10 puts questions for further study.

## 2 Preliminaries

Let us begin with some set-theoretic and geometric notation.

For a set $V \subseteq \mathbb{R}^n$, $\mathbf{I}V$ denotes its interior, $\mathbf{C}V$ denotes its closure.
$U \Subset V := U \subseteq \mathbf{I}V$ ($U$ is a non-tangential proper part of $V$),
$U \subset V := U \subseteq V$ and $U \neq V$.

Points in $\mathbb{R}^n$ are denoted by $X, Y, Z, X_1 \ldots$

We use different projections $\mathbb{R}^n \longrightarrow \mathbb{R}^{n-1}$, $\mathbb{R}^n \longrightarrow \mathbb{R}$; for $X = (x_1, \ldots, x_n)$ let
$\pi(X) := (x_2, \ldots, x_n)$, $pr_i(X) := x_i$,
$s(X) := (x_1, \ldots, x_{n-1})$; $t(X) := pr_n(X)$.

Also let
$\mathbb{R}^n_+ := \{X \in \mathbb{R}^n \mid pr_n(X) \geq 0\}$,
$\mathbb{R}^n_- := \{X \in \mathbb{R}^n \mid pr_n(X) \leq 0\}$,
$\mathbb{R}^n_0 := \{X \in \mathbb{R}^n \mid pr_n(X) = 0\}$ $(= \mathbb{R}^{n-1} \times \{0\})$.

$\|X - Y\|$ denotes the (Euclidean) distance between $X = (x_1, \ldots, x_n)$ and $Y = (y_1, \ldots, y_n)$, i.e. $\|X - Y\| := \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$. For $r \geq 0$

$$B(U, r) := \{X \mid \text{ for some } Y \in U \ \|X - Y\| \ \leq r\}$$

is the closed $r$-neighbourhood of $U \subseteq \mathbb{R}^n$; $B(X, r) := B(\{X\}, r)$ is the closed ball with center $X$ of radius $r$.

$PV = \{p_1, p_2, \dots\}$ denotes the countable set of propositional variables; as usual, modal formulas are built from $PV$, classical connectives, and modal (unary) connectives. *$n$-formulas* are built using only variables from the set $PV\lceil n := \{p_1, \dots, p_n\}$.

In this paper we consider normal modal propositional logics (as sets of formulas). For a modal logic $\mathbf{\Lambda}$ and a modal formula $A$, the notation $\mathbf{\Lambda} \vdash A$ means $A \in \mathbf{\Lambda}$; $\mathbf{\Lambda} + A$ denotes the smallest modal logic containing $\mathbf{\Lambda} \cup \{A\}$. $\mathbf{\Lambda}\lceil n$ denotes the set of all $n$-formulas in a modal logic $\mathbf{\Lambda}$.

Recall that a *(Kripke) frame* is a pair $(W, R)$, where $W$ is a non-empty set, $R$ is a binary relation on $W$. The notation $x \in F$ means $x \in W$.

A *(Kripke) model* is a Kripke frame with a valuation: $M = (W, R, \theta)$, $\theta : PV \longrightarrow 2^W$. The sign $\vDash$ denotes the truth at a point of a Kripke model and also the validity in a Kripke frame. $\mathbf{L}(F)$ denotes the set of all formulas that are valid in $F$. $F \sim_n G$ if $\mathbf{L}(F)\lceil n = \mathbf{L}(G)\lceil n$.

The notation $f : F_1 \twoheadrightarrow F_2$ means that $f$ is a p-morphism from $F_1$ onto $F_2$, and $F_1 \twoheadrightarrow F_2$ means that $f : F_1 \twoheadrightarrow F_2$ for some $f$. Recall that $F_1 \twoheadrightarrow F_2$ implies $\mathbf{L}(F_1) \subseteq \mathbf{L}(F_2)$ (P-morphism Lemma).

As usual, for a relation $R$ we denote $R(x) := \{y \mid xRy\}$. $R^\star$ denotes the transitive closure of $R$; $Id_W$ denotes the equality relation on $W$. We also put $W^x := \{x\} \cup R^\star(x)$, $F^x := (W^x, R \cap (W^x \times W^x))$ is the *subframe of $F$ generated by $x$*; recall that $\mathbf{L}(F) \subseteq \mathbf{L}(F^x)$.

A frame $F = (W, R)$ (and the relation $R$) is called *pretransitive* if for some $l$, $R^{l+1} \subseteq Id_W \cup R \cup R^2 \cdots \cup R^l$ and thus $R^\star \subseteq Id_W \cup R \cup R^2 \cdots \cup R^l$.

For a relation $R \subseteq W \times W$ let $R^\pm := R \cup R^{-1} \cup Id_W$ (R-comparability), $-R := (W \times W) - R$, $R^{\bowtie} := -(R^\pm)$ ($R$-incomparability).

Let us recall some first-order properties of a relation $R$:

| | |
|---|---|
| *seriality* | $\forall x \exists y\ xRy$; |
| *McKinsey property* | $\forall x \exists y \in R(x)\ R(y) = \{y\}$; |
| *irreflexive McKinsey property* | $\forall x (R(x) \neq \varnothing \to \exists y \in R(x)\ R(y) = \varnothing)$; |
| *density* | $\forall x \forall y \exists z (xRy \to xRz \wedge zRy)$; |
| *2-density* | $\forall x \forall y_1 \forall y_2 \exists z (xRy_1 \wedge xRy_2 \to xRz \wedge zRy_1 \wedge zRy_2)$; |
| *confluence, or Church – Rosser property* | $\forall x \forall y_1 \forall y_2 \exists z (xRy_1 \wedge xRy_2 \to y_1Rz \wedge y_2Rz)$. |

Some modal axioms and the corresponding properties of frames:

| | |
|---|---|
| $A4 := \Diamond\Diamond p \to \Diamond p$ | transitivity, |
| $AT := p \to \Diamond p$ | reflexivity, |
| $AD := \Diamond\top$ | seriality, |
| $A1 := \Box\Diamond p \to \Diamond\Box p$ | McKinsey property (for transitive frames), |
| $A1^\flat := \Diamond\top \to \Diamond\Box\bot$ | irreflexive McKinsey property, |
| $Ad = Ad_1 := \Diamond p \to \Diamond\Diamond p$ | density, |
| $Ad_2 := \Diamond p_1 \wedge \Diamond p_2 \to \Diamond(\Diamond p_1 \wedge \Diamond p_2)$ | 2-density, |
| $A2 := \Diamond\Box p \to \Box\Diamond p$ | confluence. |

We use specific notation for some modal logics:

$$\mathbf{K4} := \mathbf{K} + A4, \quad \mathbf{D4} := \mathbf{K4} + AD, \quad \mathbf{S4} := \mathbf{K4} + AT,$$

$$\mathbf{OI} := \mathbf{D4} + Ad_2, \quad \mathbf{CI} := \mathbf{K4} + Ad_2 + A1^\flat.$$

For a logic $\mathbf{\Lambda}$ let $\mathbf{\Lambda.1} := \mathbf{\Lambda} + A1$, $\mathbf{\Lambda.2} := \mathbf{\Lambda} + A2$.

## 3 Causal and chronological modalities in Minkowski spacetime

It is well-known that relativistic time is branching, due to the finiteness of the speed of light. An event (a point in a spacetime) $X$ is earlier than an event $Y$ if a signal can be sent from $X$ to $Y$. So future events may be incomparable if they are too distant in space. Bimodal temporal logics of these branching structures are still unknown, but there are some results on monomodal logics.

Recall that Minkowski metrics in $\mathbb{R}^n$, $n \geq 2$ is obtained from the following quadratic form:

$$\mu(X) := t(X)^2 - \|s(X)\|^2$$

*Chronological accessibility* $\prec$ and *causal accessibility* $\preceq$ in Minkowski spacetime are defined as follows:

$$X \prec Y \text{ iff } \mu(X - Y) > 0 \ \&\ t(Y) > t(X) \text{ iff } t(Y) - t(X) > \|s(Y) - s(X)\|,$$
$$X \preceq Y \text{ iff } \mu(X - Y) \geq 0 \ \&\ t(Y) \geq t(X) \text{ iff } t(Y) - t(X) \geq \|s(Y) - s(X)\|.$$

In the simplest cases modal logics of frames with these accessibility relations are already known.

**Theorem 1.** *[12], [27]* $\mathbf{L}(\mathbb{R}^n, \preceq) = \mathbf{S4.2}$ *for any* $n \geq 2$.

**Theorem 2.** *[26]* $\mathbf{L}(\mathbb{R}^n, \prec) = \mathbf{OI.2}$ *for any* $n \geq 2$.

**Theorem 3.** *[27] Let $U$ be an open domain in $\mathbb{R}^2$ bounded by a closed simple differentiable curve. Then* $\mathbf{L}(U, \preceq) = \mathbf{S4}$, $\mathbf{L}(\mathbf{C}U, \preceq) = \mathbf{S4.1}$.

**Theorem 4.** *[26] Let $U$ be an open domain in $\mathbb{R}^2$ bounded by a closed smooth curve. Then* $\mathbf{L}(U, \prec) = \mathbf{OI}$.

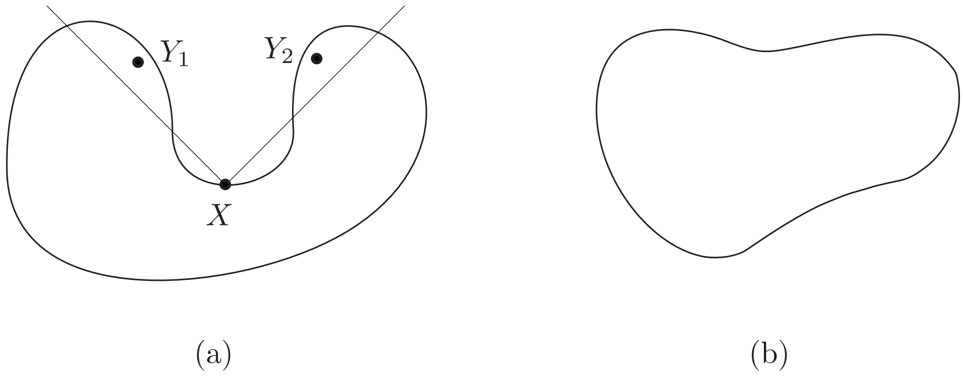



Figure 1.

Apparently Theorem 4 can be extended to the same type of domains as in Theorem 3, and similar results can be proved for higher dimensions.

For frames $F = (\mathbf{C}U, \prec)$, where $U \subset \mathbb{R}^2$ is an open domain with a smooth boundary, the situation is more delicate. In [25] it is proved that $\mathbf{L}(F) \subseteq \mathbf{CI}$. On the other hand, $F \models A4$, $F \models A1^\flat$; so $F \models Ad_2$ implies $\mathbf{L}(F) = \mathbf{CI}$. But 2-density does not always hold for frames of this kind, e.g. for the frame on Figure 1a.

Nevertheless for convex domains we have:

**Theorem 5.** *[25] Let $U$ be an open convex domain in $\mathbb{R}^2$ bounded by a closed smooth curve. Then* $\mathbf{L}(\mathbf{C}U, \prec) = \mathbf{CI}$.

On the other hand, note that the frame on Figure 1b is not convex, but 2-dense, so its logic still equals **CI**.

Let $\succeq := \preceq^{-1}$, $\succ := \prec^{-1}$. The above results imply the following

**Lemma 6.** *For $n \geq 2$ we have:*
$\mathbf{L}(\mathbb{R}^n_-, \succeq) = \mathbf{L}(\mathbf{I}\mathbb{R}^n_-, \succeq) = \mathbf{S4.2}$*,*
$\mathbf{L}(\mathbb{R}^n_-, \preceq) = \mathbf{S4.1}$, $\mathbf{L}(\mathbf{I}\mathbb{R}^n_-, \preceq) = \mathbf{S4}$*;*

$\mathbf{L}(\mathbb{R}^n_-, \succ) = \mathbf{L}(\mathbf{I}\mathbb{R}^n_-, \succ) = \mathbf{OI.2}$,

$\mathbf{L}(\mathbb{R}^n_-, \prec) = \mathbf{CI}$, $\mathbf{L}(\mathbf{I}\mathbb{R}^n_-, \prec) = \mathbf{OI}$.

Logics of polygons on Minkowski plane are also studied in [27], [26], [25]. In particular, the following holds.

**Theorem 7.** *For a convex open polygon $X \subset \mathbb{R}^2$ there may be two options:*

*(1)* $\mathbf{L}(X, \preceq) = \mathbf{S4}$, $\mathbf{L}(\mathbf{C}X, \preceq) = \mathbf{S4.1}$, $\mathbf{L}(X, \prec) = \mathbf{OI}$;

*(2)* $\mathbf{L}(X, \preceq) = \mathbf{S4.2}$, $\mathbf{L}(\mathbf{C}X, \preceq) = \mathbf{S4.1.2}$, $\mathbf{L}(X, \prec) = \mathbf{OI.2}$.

Let us also mention the correlation between product logics [10], [9] and relativistic logics. Consider the product frame

$(\mathbb{R}, <)^2 := (\mathbb{R}, <) \times (\mathbb{R}, <) := (\mathbb{R}^2, <_1, <_2)$,

where

$(x, y) <_1 (x', y') \Leftrightarrow x < x' \;\&\; y = y'$;

$(x, y) <_2 (x', y') \Leftrightarrow x = x' \;\&\; y < y'$.

Relativistic time can be interpreted within this frame. In fact, the frames $(\mathbb{R}^2, <_1 \circ <_2)$ and $(\mathbb{R}^2, \prec)$ are isomorphic (by rotation). So the logic $\mathbf{L}(\mathbb{R}^2, \prec) = \mathbf{OI}$ is naturally embedded into $\mathbf{L}((\mathbb{R}, <)^2)$:

**Corollary 8.** $\mathbf{OI.2} = \{A \mid (\mathbb{R}, <)^2 \vDash \varphi(A)\}$,
*where $\varphi$ translates $\Box$ as $\Box_1\Box_2$ and does not affect other connectives.*

Note that the whole bimodal logic $\mathbf{L}((\mathbb{R}, <)^2)$ is $\Pi^1_1$-complete, and therefore not recursively axiomatizable [9].

Similarly we have

**Corollary 9.** $\mathbf{S4.2} = \{A \mid (\mathbb{R}, \leq)^2 \vDash \varphi(A)\}$, *where $\varphi$ is the same as in Corollary 8.*

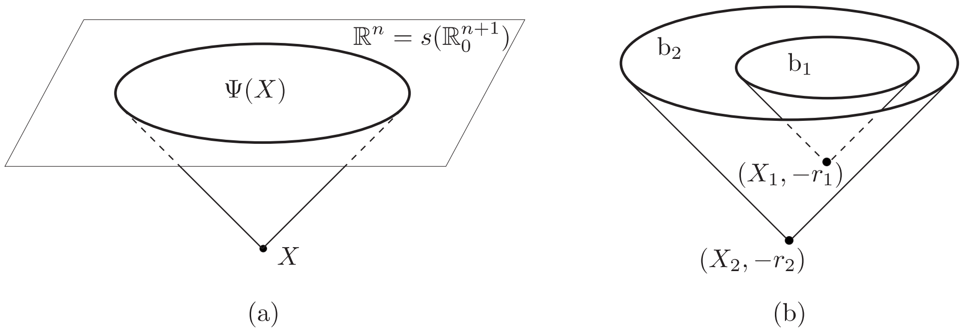



Figure 2.

# 4 Balls and intervals

In this Section we interpret the previous results in terms of balls. This can be done due to a natural correspondence between points in Minkowski $(n+1)$-half-space and $n$-balls.

Let

$$\mathcal{B}_n := \{B(X,r) \mid X \in \mathbb{R}^n, r > 0\},\ \mathcal{B}^*_n := \{B(X,r) \mid X \in \mathbb{R}^n, r \geq 0\}.$$

There is a standard bijection $\Psi : \mathbb{R}^{n+1}_- \longrightarrow \mathcal{B}^*_n$ such that

$$\Psi(X) := s(\preceq (X) \cap \mathbb{R}^{n+1}_0)$$

(Fig. 2a). The other way round, for $\mathrm{b} = B(X,r) \in \mathcal{B}^*_n$, we have $\Psi^{-1}(\mathrm{b}) = (X, -r) \in \mathbb{R}^{n+1}_-$.

**Definition 10.** For a relation $R$ on $\mathcal{B}^*_n$, its *lower-correspondent* is the relation $S$ on $\mathbb{R}^{n+1}_-$ such that $\Psi$ is an isomorphism between $(\mathcal{B}^*_n, R)$ and $(\mathbb{R}^{n+1}_-, S)$; notation: $R \xleftarrow{\Psi} S$.

**Lemma 11.** $\supseteq \xleftarrow{\Psi} \preceq$, $\Supset \xleftarrow{\Psi} \prec$, $\subseteq \xleftarrow{\Psi} \succeq$, $\Subset \xleftarrow{\Psi} \succ$.

**Proof.** In fact, for balls $b_1 = B(X_1, r_1)$, $b_2 = B(X_2, r_2)$ we have

$$b_1 \subseteq b_2 \text{ iff } r_2 - r_1 \geq \|X_1 - X_2\| \text{ iff } (X_2, -r_2) \preceq (X_1, -r_1)$$

(Fig. 2b), thus $\subseteq \overset{\Psi}{\longleftarrow} \succeq$, $\supseteq \overset{\Psi}{\longleftarrow} \preceq$. Two other claims are proved in a similar way. ■

From Lemmas 6, 11 we obtain

**Theorem 12.** *Completeness results for logics of $n$-dimensional balls, $n \geq 1$ are presented in Table 1.*

Table 1. Logics of balls in $\mathbb{R}^n$

| | $\Supset$ | $\supseteq$ | $\Subset$ | $\subseteq$ |
|---|---|---|---|---|
| $\mathcal{B}_n$ | **OI** | **S4** | **OI.2** | **S4.2** |
| $\mathcal{B}_n^*$ | **CI** | **S4.1** | **OI.2** | **S4.2** |

For the case $n = 1$, $\mathcal{B}_1 = \mathcal{I}$, $\mathcal{B}_1^* = \mathcal{I}^*$ are the sets of all strict and non-strict closed intervals on the real line:

$$\mathcal{I} := \{[a, b] \mid a < b\},\ \mathcal{I}^* = \{[a, b] \mid a \leq b\}.$$

So Theorem 12 also describes logics of intervals with strict and non-strict subinterval relations:

$$[a_1, b_1] \subseteq [a_2, b_2] \text{ iff } a_2 \leq a_1 \ \&\ b_1 \leq b_2,$$
$$[a_1, b_1] \Subset [a_2, b_2] \text{ iff } a_2 < a_1 \ \&\ b_1 < b_2.$$

## 5 Regions and bricks in $\mathbb{R}^n$

There are different options for spatial analogues of intervals; the corresponding modal logics are often undecidable (see Section 8 below). But in the simplest cases we obtain the same modal logics as in the previous Section.

Recall that a closed set $V$ is called *regular* if $\mathbf{CI}V = V$. Let $\mathcal{C}N_n$ (respectively, $\mathcal{C}V_n$) be the set of all non-empty compact regular sets with the connected (respectively, convex) interior in $\mathbb{R}^n$, and let $\mathcal{C}N_n^*$ ($\mathcal{C}V_n^*$) be its extension by all singletons. Sets of all these types are called *regions*. $n$*-dimensional bricks* are a special type of regions:

$$\mathcal{R}_n := \{\prod_{i=1}^{n} \mathrm{u}_i \mid \mathrm{u}_i \in \mathcal{I}\}, \quad \mathcal{R}_n^* := \{\prod_{i=1}^{n} \mathrm{u}_i \mid \mathrm{u}_i \in \mathcal{I}^*\}.$$

Trivially, $\mathcal{C}N_1 = \mathcal{C}V_1 = \mathcal{R}_1 = \mathcal{I}$ and $\mathcal{C}N_1^* = \mathcal{C}V_1^* = \mathcal{R}_1^* = \mathcal{I}^*$.

**Theorem 13.** *Table 2 contains completeness results for some logics of regions and bricks, $n \geq 1$.*

Table 2. Logics of regions and bricks

| | $\Supset$ | $\supseteq$ | $\Subset$ | $\subseteq$ |
|---|---|---|---|---|
| $\mathcal{C}V_n, \mathcal{C}N_n, \mathcal{R}_n$ | **OI** | **S4** | **OI.2** | **S4.2** |
| $\mathcal{C}V_n^*, \mathcal{C}N_n^*, \mathcal{R}_n^*$ | **CI** | **S4.1** | **OI.2** | **S4.2** |

**Proof.**

(Soundness.) It is clear that all four relations are transitive; $\subseteq$, $\supseteq$ are reflexive, and $\Subset$ is serial. It is also clear that $\Supset$ is serial if singletons are not involved. To check the confluence of $\subseteq$, $\Subset$, note that the union of two compact sets is compact, and every compact set can be covered by a brick (which is in $\mathcal{CV}^*_n$, $\mathcal{CN}^*_n$ as well).

To show that $(\mathcal{R}_n, \Supset) \models Ad_2$, consider bricks $\mathrm{r} = \prod_{i=1}^{n} \mathrm{u}_i$, $\mathrm{r}' = \prod_{i=1}^{n} \mathrm{u}'_i$, $\mathrm{r}'' = \prod_{i=1}^{n} \mathrm{u}''_i$, where $\mathrm{u}_i, \mathrm{u}'_i, \mathrm{u}''_i \in \mathcal{I}$, and suppose $\mathrm{r} \Supset \mathrm{r}'$, $\mathrm{r} \Supset \mathrm{r}''$. Since $(\mathcal{I}, \Supset)$ is 2-dense, for some $\mathrm{v}_1, \ldots, \mathrm{v}_n$ we have: $\mathrm{u}_i \Supset \mathrm{v}_i \Supset \mathrm{u}'_i$; $\mathrm{v}_i \Supset \mathrm{u}''_i$. Then for $\mathrm{s} := \prod_{i=1}^{n} \mathrm{v}_i$, we obtain: $\mathrm{r} \Supset \mathrm{s} \Supset \mathrm{r}'$; $\mathrm{s} \Supset \mathrm{r}''$. Therefore $(\mathcal{R}_n, \Supset) \models Ad_2$.

Since $(\mathcal{R}^*_n, \Supset)$ contains singletons, which are dead ends, it follows that $(\mathcal{R}^*_n, \Supset) \models A1^\flat$. Thus $\mathbf{L}(\mathcal{R}^*_n, \Supset) \supseteq \mathbf{CI}$.

Let us prove the 2-density of $\Subset$ for $\mathcal{CN}_n$ and $\mathcal{CV}_n$. Let $\mathrm{v}, \mathrm{u}_1, \mathrm{u}_2 \in \mathcal{CN}_n$, $\mathrm{v} \Subset \mathrm{u}_1$, $\mathrm{v} \Subset \mathrm{u}_2$. Then $\mathrm{v} \Subset \mathrm{u}_1 \cap \mathrm{u}_2$, and so for some $r > 0$ we have $B(\mathrm{v}, r) \Subset \mathrm{u}_1 \cap \mathrm{u}_2$. Since $\mathrm{v} \Subset B(\mathrm{v}, r)$ and $B(\mathrm{v}, r) \in \mathcal{CN}_n$, it follows that $(\mathcal{CN}_n, \Supset) \models Ad_2$. If $\mathrm{v} \in \mathcal{CV}_n$, then $B(\mathrm{v}, r) \in \mathcal{CV}_n$ as well, so $(\mathcal{CV}_n, \Supset) \models Ad_2$.

Similarly one can prove the 2-density of $\Supset$ for $\mathcal{CN}_n$ and $\mathcal{CV}_n$.

(Completeness.) Let $d(V)$ be the direct image of $V \subseteq \mathbb{R}^n$ under $pr_1$. Since continuous maps preserve compactness and connectedness, it follows that $d(V) \subset \mathbb{R}$ is compact and connected, i.e. $d(V) \in \mathcal{I}^*$. One can easily see that $d(V) \in \mathcal{I}$ whenever $V \in \mathcal{CN}_n$ and moreover, for any $W \in \{\mathcal{CN}_n, \mathcal{CV}_n, \mathcal{R}_n\}$, $R \in \{\Supset, \supseteq, \Subset, \subseteq\}$,

$$d : (W, R) \twoheadrightarrow (\mathcal{I}, R),\ d : (W^*, R) \twoheadrightarrow (\mathcal{I}^*, R).$$

Hence by Theorem 12, we obtain completeness. ■

# 6 Remarks on "After"

Monomodal logics based on the relations $\subseteq$, $\Subset$ and their spacetime analogues $\prec$, $\preceq$, do not allow us either to determine the dimension of space, or to distinguish balls from bricks. But R. Goldblatt noticed that the modality "after" might be more expressive. It corresponds to relation $\alpha$, the simplest irreflexive version of $\preceq$, i.e. for $X, Y \in \mathbb{R}^n$, $n \geq 2$,

$$X \alpha Y := X \preceq Y \text{ and } X \neq Y.$$

It is easy to see that $\alpha$ is transitive, dense, serial, and confluent, but not 2-dense. (To disprove the 2-density for $(\mathbb{R}^2, \alpha)$, take the points $(0,0)$, $(-1,1)$, $(1,1)$.) But some subtler modally definable properties still hold for $\alpha$.

First consider the formulas

$$Ad_{n,2} := \bigwedge_{1 \leq i \leq n} \Diamond p_i \to \bigvee_{1 \leq i < j \leq n} \Diamond(\Diamond p_i \wedge \Diamond p_j)$$

(as usual, we assume that $\bigwedge_{i \in \varnothing} A_i = \top$, $\bigvee_{i \in \varnothing} A_i = \bot$). These are Sahlqvist formulas; the first-order correspondent of $Ad_{n,2}$ is

$$\forall x \forall x_1 \ldots \forall x_n \left( \bigwedge_{1 \leq i \leq n} x R x_i \to \bigvee_{1 \leq i < j \leq n} \exists y (x R y \wedge y R x_i \wedge y R x_j) \right).$$

**Lemma 14.**

*(1)* $\mathbf{K} + Ad_{n,2} \vdash Ad_{n+1,2}$*,*

*(2)* $\mathbf{K} + Ad_{n,2} \vdash Ad_1$*.*

**Proof.** (1) is almost obvious. To show (2), substitute $p$ for every $p_i$ in $Ad_{n,2}$. ■

**Lemma 15.** *[12]*

*(1)* $(\mathbb{R}^2, \alpha) \vDash Ad_{3,2}$*.*

*(2)* $(\mathbb{R}^3, \alpha) \nvDash Ad_{n,2}$ *for any* $n$*.*

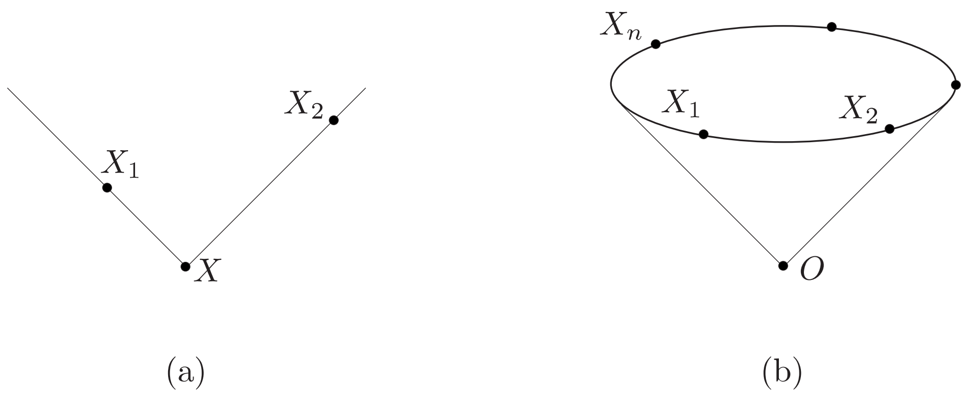



Figure 3.

**Proof.** (1) Suppose $X\alpha X_1$, $X\alpha X_2$, but there is no $Y$ such that $X\alpha Y$, $Y\alpha X_1$, $Y\alpha X_2$. It follows that $X_1$, $X_2$ are on two different sides of the light-cone $\preceq(X)$, see Fig. 3a. But then $X\alpha X_3$ implies that either the corresponding $Y$ exists for $X_1$ and $X_3$, or $X_3$ and $X_2$ are on the same side of the cone (and so we can take $Y$ on this side as well).

(2) Take different points $X_1, \ldots, X_n$ on the circle

$$\begin{cases} x^2 + y^2 = 1 \\ z = 1 \end{cases}$$

Every $X_i$ is seen from $O = (0,0,0)$ (Fig. 3b), but $O$ is the maximal point seeing two distinct $X_i$s. Thus $Ad_{n,2}$ fails. ■

Now let us consider another extra axiom

$$\begin{aligned} Aaf := \quad & \Diamond(\Diamond(p_1 \wedge \neg p_2 \wedge \Box\neg p_2) \wedge \Diamond(p_2 \wedge \neg p_1 \wedge \Box\neg p_1)) \wedge \Diamond q \to \\ & \Diamond(\Diamond p_1 \wedge \Diamond q) \vee \Diamond(\Diamond p_2 \wedge \Diamond q), \end{aligned}$$

and let

$$\mathbf{L}\alpha_0 := \mathbf{D4.2} + Ad + Aaf.$$

This axiom corresponds to the following first-order condition:

$$\forall x \forall y \forall y_1 \forall y_2 [(xRy \wedge yRy_1 \wedge yRy_2 \wedge y_1 R^{\bowtie} y_2 \rightarrow \\ \forall z \exists t (xRz \rightarrow xRt \wedge tRz \wedge (tRy_1 \vee tRy_2))].$$

**Proposition 16.**

*(1)* $\mathbf{L}(\mathbb{R}^3, \alpha) \subset \mathbf{L}(\mathbb{R}^2, \alpha)$

*(2)* $\mathbf{L}\alpha_0 \subseteq \mathbf{L}(\mathbb{R}^{n+1}, \alpha) \subseteq \mathbf{L}(\mathbb{R}^n, \alpha)$ *for* $n \geq 3$

**Proof.** We have $\pi : (\mathbb{R}^{n+1}, \alpha) \twoheadrightarrow (\mathbb{R}^n, \alpha)$, so $\mathbf{L}(\mathbb{R}^{n+1}, \alpha) \subseteq \mathbf{L}(\mathbb{R}^n, \alpha)$. By Lemma 15, $\mathbf{L}(\mathbb{R}^3, \alpha) \neq \mathbf{L}(\mathbb{R}^2, \alpha)$.

It remains to show the validity of $Aaf$ in $(\mathbb{R}^n, \alpha)$. Let us explain this for the case $n = 3$, the general case is quite similar — just replace 2-disks with $(n-1)$-disks. So suppose $X\alpha Y$, $Y\alpha Y_1$, $Y\alpha Y_2$, $X\alpha Z$ and $Y_1, Y_2$ are $\alpha$-incomparable. Take the cones from these points and consider their section by a sufficiently high plane. The section of every cone is a disk on this plane; let us denote these disks by the same letters $X, Y, \dots$ (see Fig. 4). Now we have to find a disk $T \subset X$ containing $Z$ and either $Y_1$ or $Y_2$. If $Y \Subset X$ or $Z \Subset X$, then such a covering disk exists already for $Z$ and $Y$. Otherwise, both $Y$, $Z$ are internally tangent to $X$. Next, if $Y_1$ is also internally tangent to $X$, then $Y_2$ is not (since $Y_1$, $Y_2$ are incomparable); then a covering disk $T$ exists for $Y_2$ and $Z$. ∎

Similarly we obtain

**Proposition 17.**

*(1)* $\mathbf{L}(\mathcal{B}_1, \subset) \supset \mathbf{L}(\mathcal{B}_2, \subset) \supseteq \mathbf{L}(\mathcal{B}_3, \subset) \supseteq \cdots \supseteq \mathbf{L}\alpha_0$

*(2)* $\mathbf{L}(\mathcal{R}_m, \subset) = \mathbf{L}(\mathcal{R}_n, \subset)$ *iff* $m = n$.

*(3)* $\mathbf{L}(\mathcal{B}_m, \subset) = \mathbf{L}(\mathcal{R}_n, \subset)$ *iff* $m = n = 1$.

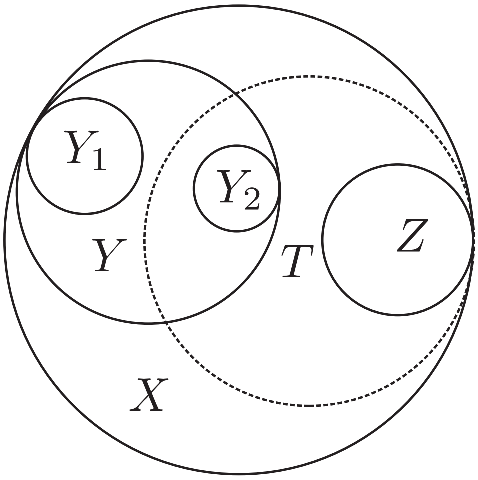



Figure 4.

**Proof.**

(1) Trivially, $\mathbf{L}(\mathbb{R}^n, \alpha) = \mathbf{L}(\mathbb{R}^n_-, \alpha^{-1})$ for any $n \geq 2$.

Since $\subset \overset{\Psi}{\longleftarrow} \alpha^{-1}$, by Proposition 16, we obtain (1).

(2) Let us show that $(\mathcal{R}_n, \subset) \vDash Ad_{2n+1,2}$. Consider bricks $\mathrm{r}, \mathrm{r}_1, \ldots, \mathrm{r}_{2n+1}$ such that $\mathrm{r} \subset \mathrm{r}_i$ for any $i$, and assume that $\mathrm{r} = \prod\limits_{j=1}^{n} \mathrm{v}_j$, $\mathrm{r}_i = \prod\limits_{j=1}^{n} \mathrm{v}_{ij}$, where $\mathrm{v}_i, \mathrm{v}_{ij} \in \mathcal{I}$. Then for any $i$, $j$ $(1 \leq i \leq 2n+1,\ 1 \leq j \leq n)$, $\mathrm{v}_j \subseteq \mathrm{v}_{ij}$, and for any $i$ there exists $j_i$ such that $\mathrm{v}_{j_i} \subset \mathrm{v}_{ij_i}$. By pigeonhole principle, at least three of these numbers $j_i$ coincide. So for some $j$ and for some distinct $i_1, i_2, i_3$ we have $\mathrm{v}_j \subset \mathrm{v}_{i_1 j}$, $\mathrm{v}_j \subset \mathrm{v}_{i_2 j}$, $\mathrm{v}_j \subset \mathrm{v}_{i_3 j}$. Therefore, since $(\mathcal{I}, \subset) \vDash Ad_{3,2}$, it follows that $(\mathcal{R}, \subset) \vDash Ad_{2n+1,2}$.

On the other hand, a straightforward argument shows that $(\mathcal{R}_n, \subset) \nvDash Ad_{2n,2}$, so by Lemma 14, $(\mathcal{R}_n, \subset) \nvDash Ad_{l,2}$ for any $l \leq 2n$. Thus $m < n$ implies $(\mathcal{R}_n, \subset) \nvDash Ad_{2m+1,2}$, while $(\mathcal{R}_m, \subset) \vDash Ad_{2m+1,2}$.

(3) Since $\subset \overset{\Psi}{\longleftarrow} \alpha^{-1}$, by Lemma 15 we have $(\mathcal{B}_2, \subset) \nvDash Ad_{l,2}$ for any $l$; thus

by (1), $(\mathcal{B}_n, \subset) \not\models Ad_{l,2}$ for $n > 2$. ■

However we do not know exact axiomatizations for logics of frames considered in this Section; the problem of their decidability is also open.

# 7 Non-finitely axiomatizable logics

It seems that in many cases relativistic modal logics are not finitely axiomatizable; some examples are presented in this Section. Our arguments are based on the following simple fact.

**Lemma 18.** *Consider a logic $\Lambda$ and suppose that for every $n$ there exist frames $G_n$, $G'_n$ such that $G_n \sim_n G'_n$, $\Lambda \subseteq \mathbf{L}(G_n)$, $\Lambda \not\subseteq \mathbf{L}(G'_n)$. Then $\Lambda$ is not finitely axiomatizable, and moreover, not axiomatizable by any set of $n$-formulas with $n$ fixed.*

**Proof.** Almost trivial. Consider a set $\Gamma \subset \Lambda\lceil n$. Then $G_n \models \Gamma$, and $G_n \sim_n G'_n$ implies $G'_n \models \Gamma$. Since $\Lambda \not\subseteq \mathbf{L}(G'_n)$, we obtain $\Lambda \neq \mathbf{K}+\Gamma$. ■

For every finite rooted pretransitive frame $F$ one can construct an analogue of Jankov–Fine frame formula $X(F)$ (cf. [5]), with the following property:

**Lemma 19.** *Let $F$ be a finite pretransitive rooted frame, $G$ an arbitrary frame. Then*

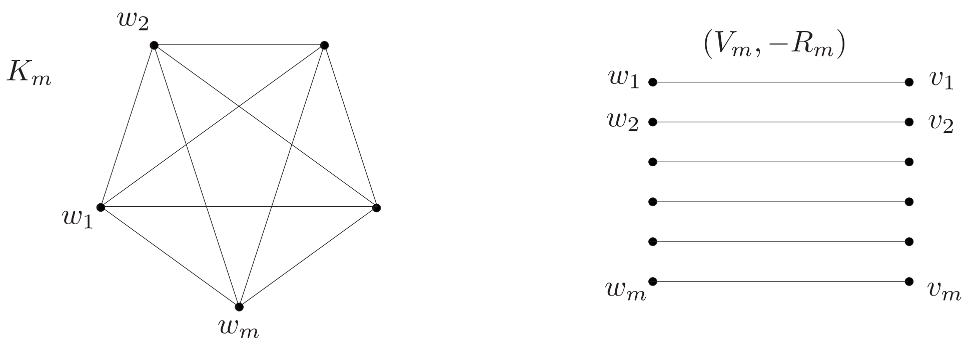
$$\mathbf{L}(G) \subseteq \mathbf{L}(F) \text{ iff } G \not\models X(F) \text{ iff for some } w \in G,\ G^w \twoheadrightarrow F.$$

Now let us define frames $K_m, K'_m, L_m, L'_m$, for $m \geq 1$ (Fig. 5):

$W_m = \{w_1, \ldots, w_m\},\ K_m = (W_m, \neq),\ K'_m := (W_m, \neq \cup\{(w_m, w_m)\});$

$V_m := \{w_1, v_1, \ldots, w_n, v_n\},\ R_m := -\{(w_i, v_i), (v_i, w_i)\}_{1 \leq i \leq m},$

$L_m := (V_m, R_m),\ L'_m := (V_m, R_m \cup \{(w_m, v_m), (v_m, w_m)\}).$




(a) (b)

Figure 5.

So $K_m$ is an "irreflexive $m$-clique".

**Lemma 20.**

*(1)* $K_{2^n+1} \sim_n K'_{2^n}$;

*(2)* $L_{2^{2n}+1} \sim_n L'_{2^{2n}}$.

**Proof.** For worlds $u$, $v$ in a certain Kripke model $M$, we write $u \equiv_0 v$ if $\forall i \leq n\,(M, u \vDash p_i \Leftrightarrow M, v \vDash p_i)$. It is clear that the equivalence relation $\equiv_0$ has at most $2^n$ classes.

(1) Consider an arbitrary model $M$ over $K_{2^n+1}$. Then for some distinct $a, b \in M$ we have: $a \equiv_0 b$. By symmetry, we may assume that $a = w_{2^n},\ b = w_{2^n+1}$. Now we merge these points into a single reflexive point, and obtain a model $M'$ over the frame $K'_{2^n}$.

By induction on the length of an $n$-formula $A$ it follows that

$$\text{for any } i < 2^n\ M, w_i \vDash A \text{ iff } M', w_i \vDash A;$$
$$M, w_{2^n} \vDash A \text{ iff } M, w_{2^n+1} \vDash A \text{ iff } M', w_{2^n} \vDash A.$$

Hence $K_{2^n+1} \sim_n K'_{2^n}$.

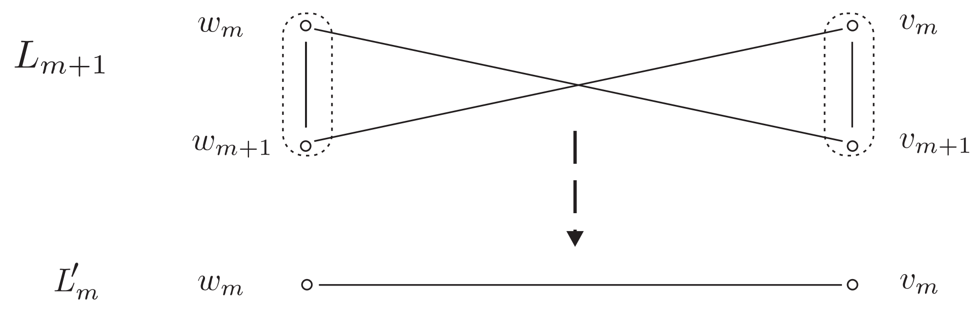



Figure 6.

(2) Consider a model $M$ over $L_{m+1}$, where $m = 2^{2n}$. There exist at most $2^{2n}$ pairs of $\equiv_0$-classes in $M$, so for some distinct $i$, $j$ we have $w_i \equiv_0 w_j$ and $v_i \equiv_0 v_j$. Without any loss of generality we may assume that $i = m$, $j = m + 1$. Then we can identify $w_m$ with $w_{m+1}$, and also $v_m$ with $v_{m+1}$, and obtain a model $M'$ over the frame $L'_m$ (Fig. 6).

By induction we obtain that for any $n$-formula $A$, for any $i < m$

$$M, w_i \vDash A \text{ iff } M', w_i \vDash A,$$
$$M, v_i \vDash A \text{ iff } M', v_i \vDash A,$$
$$M, w_m \vDash A \text{ iff } M, w_{m+1} \vDash A \text{ iff } M', w_m \vDash A,$$
$$M, v_m \vDash A \text{ iff } M, v_{m+1} \vDash A \text{ iff } M', v_m \vDash A.$$

Hence the claim follows. ■

Now consider the relations $\prec^{\pm}$ and $\preceq^{\bowtie}$. Note that for distinct $X, Y \in \mathbb{R}^n$

$X \prec^{\pm} Y$ iff the vector $(X - Y)$ is *timelike*, i.e. $\mu(X - Y) > 0$;
$X \preceq^{\bowtie} Y$ iff the vector $(X - Y)$ is *spacelike*, i.e. $\mu(X - Y) < 0$.

**Theorem 21.** *For $R \in \{\preceq^{\pm}, \preceq^{\bowtie}, \prec^{\pm}, \prec^{\bowtie}\}$ the logic $\mathbf{L}(\mathbb{R}^n, R)$, $n \geq 2$ is not finitely axiomatizable (and not axiomatizable in finitely many variables).*

**Proof.**

(I) Let $R \in \{\preceq^{\bowtie}, \prec^{\bowtie}\}$. It is sufficient to show that for any $l$, $X \in \mathbb{R}^n$

(1) $(\mathbb{R}^n, R) \twoheadrightarrow K'_l$;

(2) $(\mathbb{R}^n, R)^X \not\twoheadrightarrow K_l$.

In fact, let $\mathbf{\Lambda} = \mathbf{L}(\mathbb{R}^n, R)$. Then (1) implies $\mathbf{\Lambda} \subseteq \mathbf{L}(K'_l)$, while (2) implies $\mathbf{\Lambda} \not\subseteq \mathbf{L}(K_l)$, by Lemma 19. Hence by Lemmas 20 and 18 it follows that $\mathbf{\Lambda}$ is not axiomatizable in finitely many variables.

To prove (1), let us take distinct parallel straight lines $Q_1, \ldots, Q_{l-1}$ of timelike direction, and let us define a map $f : \mathbb{R}^n \longrightarrow K'_l$ as follows:

$$f(X) := \begin{cases} w_i & \text{if } X \in Q_i, \\ w_l & \text{otherwise.} \end{cases}$$

Then $f : (\mathbb{R}^n, R) \twoheadrightarrow K'_l$ for $R \in \{\preceq^{\bowtie}, \prec^{\bowtie}\}$. In fact, every line $Q_i$ contains points that are $\preceq^{\bowtie}$-related to $X$, whenever $X \notin Q_i$.

For the proof of (2), note that the $(\mathbb{R}^n, \preceq)$ contains arbitrarily large antichains (containing the given $X$). Thus $(\mathbb{R}^n, R)^X$ contains a subframe isomorphic to $K_{l+1}$, which cannot be mapped monotonically onto $K_l$.

(II) Now let $R \in \{\preceq^{\pm}, \prec^{\pm}, \}$. Again by Lemma 20, it is sufficient to show

(1) $(\mathbb{R}^n, R) \twoheadrightarrow L'_l$,

(2) $(\mathbb{R}^n, R)^X \not\twoheadrightarrow L_l$.

To check (1), we take different hyperplanes $P_1, \ldots, P_{l-1}$ parallel to $\mathbb{R}^n_0$: $P_i = \{X \in \mathbb{R}^n \mid t(X) = i\}$. Let us split each $P_i$ into two dense subsets, $P'_i$ and $P''_i$. Let $U := \mathbb{R}^n - (P_1 \cup \cdots \cup P_{l-1})$, $U_+ := U \cap \mathbb{R}^n_+$. Then for $X \in \mathbb{R}^n$

we put:

$$g(X) := \begin{cases} v_i & \text{if } X \in P'_i, \\ w_i & \text{if } X \in P''_i, \\ v_l & \text{if } X \in U_+, \\ w_l & \text{otherwise.} \end{cases}$$

It follows that $g : (\mathbb{R}^n, R) \twoheadrightarrow L'_l$; in fact, if $X \notin P_i$, then the geometric cone $\prec^{\pm} (X)$ intersects $P'_i$ and $P''_i$.

To prove (2), note that the frame $(\mathbb{R}^n, \prec)$ is *directed*, and thus for any $X_1, \ldots, X_k$ there exists $Y$ such that $X_1 RY, \ldots, X_k RY$. This property transfers to p-morphic images, so if $(\mathbb{R}^n, R)^X \twoheadrightarrow L_l$, then $L_l$ should contain a point related to all other points, which is a contradiction. ∎

Similarly one can prove the following

**Theorem 22.** *For any* $n \geq 1$,

$$W \in \{\mathcal{B}_n, \mathcal{R}_n, \mathcal{C}N_n, \mathcal{C}V_n, \mathcal{B}^*_n, \mathcal{R}^*_n, \mathcal{C}N^*_n, \mathcal{C}V^*_n\},$$
$$R \in \{\Subset^{\pm}, \Subset^{\bowtie}, \subseteq^{\pm}, \subseteq^{\bowtie}\},$$

*the logic* $\mathbf{L}(W, R)$ *is not finitely axiomatizable.*

# 8 Finite Model Property and Complexity

The logics **S4**, **S4.1**, **S4.2** are well-known; they all have the *finite model property* (FMP) and are PSPACE-complete, cf. [6], [5], [24].

The FMP for the logics **OI**, **OI.2** is proved in [26]; a similar method is used for **CI** in [25].

The complexity of 2-dense logics was first studied in [24], where the proof of PSPACE-completeness for **OI**, **OI.2** was given. A slight modification of this proof yields the PSPACE-completeness for **CI**.

Therefore the simplest regional, interval and spacetime logics enjoy the FMP and are PSPACE-complete. But more expressive systems turn out to be undecidable [15],[17]. Let us recall some of these results and give their analogues for relativistic logics.

Recall that RCC5-relations between regions are $\{=,\ \subset,\ \supset,\ \between,\ \asymp\}$, where

$$\mathrm{v} \between \mathrm{u} \ := \ \mathbf{I}\mathrm{v} \cap \mathbf{I}\mathrm{u} \neq \varnothing \ \&\ \mathrm{v} \subseteq^{\bowtie} \mathrm{u} \ \ \text{(meeting)},$$
$$\mathrm{v} \asymp \mathrm{u} \ := \ \mathrm{v} \cap \mathrm{u} = \varnothing \ \ \text{(partial overlapping)}.$$

Consider spacetime correspondents of $\between$, $\asymp$; viz. for $X_1, X_2 \in \mathbb{R}_-^{n+1}$ we put:

$$X_1 \between' X_2 := X_1 \preceq^{\bowtie} X_2 \text{ and for some } Y \in \mathbb{R}_-^{n+1}\ X_1 \prec Y \text{ and } X_2 \prec Y,$$
$$X_1 \asymp' X_2 := \text{ there is no } Y \in \mathbb{R}_0^{n+1} \text{ such that } X_1 \preceq Y \text{ and } X_2 \preceq Y.$$

One can see that $\between \xleftarrow{\Psi} \between'$, $\asymp \xleftarrow{\Psi} \asymp'$, so we obtain

**Proposition 23.**
$\mathbf{L}(\mathbb{R}_-^{n+1}, \alpha, \alpha^{-1}, \between', \asymp') = \mathbf{L}(\mathcal{B}_n^*, \subset, \supset, \between, \asymp)$;
$\mathbf{L}(\mathbf{I}\mathbb{R}_-^{n+1}, \alpha, \alpha^{-1}, \between', \asymp') = \mathbf{L}(\mathcal{B}_n, \subset, \supset, \between, \asymp)$.

In [17] it proved that every logic $\mathbf{L}(\mathcal{R}_n, \subset, \supset, \between, \asymp)$ is undecidable. Since $\mathcal{R}_1 = \mathcal{B}_1$, by Proposition 23 we obtain

**Proposition 24.** *The logic* $\mathbf{L}(\mathbf{I}\mathbb{R}_-^2, \alpha, \alpha^{-1}, \between', \asymp')$ *is undecidable.*

## 9 Remarks on intuitionistic logic

It is well-known that every intuitionistic formula $A$ can be transformed into a modal formula $T(A)$ via Gödel – Tarski translation (putting $\Box$ in front of every subformula). Thus every (consistent) modal logic $\mathbf{\Lambda}$ above **S4** corresponds to an intermediate logic $s(\mathbf{\Lambda}) := \{A \mid T(A) \in \mathbf{\Lambda}\}$ (the *superintuitionistic fragment* of $\mathbf{\Lambda}$). For a Kripke frame $F = (W, R)$ with $R$ transitive and reflexive, we obtain *the intermediate logic* of $F$: $\mathbf{IL}(F) := s(\mathbf{L}(F))$.

It is also well-known that the intuitionistic logic $\mathbf{H}$ is $s(\mathbf{S4}) = s(\mathbf{S4.1})$, and $s(\mathbf{S4.2}) = s(\mathbf{S4.2.1}) = \mathbf{H} + \neg p \vee \neg\neg p$ (the logic of the weak excluded middle denoted by **KC**).

So we have the following consequence from Sections 3, 4, 5.

**Corollary 25.**

*(1)* $\mathbf{IL}(\mathbb{R}^n, \preceq) = \mathbf{KC}$ *for any* $n \geq 2$.

*(2) Let* $U$ *be an open domain in* $\mathbb{R}^2$ *bounded by a closed simple differentiable curve. Then* $\mathbf{IL}(U, \preceq) = \mathbf{IL}(\mathbf{C}U, \preceq) = \mathbf{H}$.

*(3) For a convex open polygon* $X \subset \mathbb{R}^2$,

$\mathbf{IL}(X, \preceq) = \mathbf{IL}(\mathbf{C}X, \preceq)$ *is either* $\mathbf{H}$ *or* $\mathbf{KC}$.

*(4) For* $W \in \{\mathcal{B}_n, \mathcal{R}_n, \mathcal{C}N_n, \mathcal{C}V_n, \mathcal{B}^*_n, \mathcal{R}^*_n, \mathcal{C}N^*_n, \mathcal{C}V^*_n\}$ *we have* $\mathbf{IL}(W, \supseteq) = \mathbf{H}$, $\mathbf{IL}(W, \subseteq) = \mathbf{KC}$.

The frames mentioned in (4) also admit interpretation using Medvedev's notion of "information types" [19]. Let us briefly describe it in an equivalent Kripke-style form.

A region can be regarded as "information" about some unknown point in this region (in particular, an interval gives information about a real number). The inclusion u $\subseteq$ v means that information u "refines" v. The truth value of an intuitionistic proposition depends on the information we have; u $\Vdash A$ (u "forces" $A$ ) if u is sufficient for stating $A$. Of course, if u $\supseteq$ v and u $\Vdash A$, it should be that v $\Vdash A$. Given truth values of basic propositions (an *intuitionistic valuation*), we can find truth values of all intuitionistic formulas according to the standard rules:

u $\Vdash A \wedge B$ iff u $\Vdash A$ and u $\Vdash B$;

u $\Vdash A \vee B$ iff u $\Vdash A$ or u $\Vdash B$;

u $\Vdash \neg A$ iff v $\nVdash A$ for any v $\subseteq$ u;

u $\Vdash A \rightarrow B$ iff v $\Vdash A$ implies v $\Vdash B$ for any v $\subseteq$ u.

As usual, a formula $A$ is called *valid* in $(W, \supseteq)$ (notation: $(W, \supseteq) \Vdash A$) if every information forces $A$ under any intuitionistic valuation. A standard argument shows that the validity of $A$ is equivalent to the validity of the modal formula $T(A)$ in the corresponding Kripke frame. This implies the following reformulation of Corollary 25 (4):

**Proposition 26.** *For* $W \in \{\mathcal{B}_n, \mathcal{R}_n, \mathcal{C}N_n, \mathcal{C}V_n, \mathcal{B}^*_n, \mathcal{R}^*_n, \mathcal{C}N^*_n, \mathcal{C}V^*_n\}$ *we have* $(W, \supseteq) \Vdash A$ *iff* $\mathbf{H} \vdash A$.

In [19] validity is defined in terms of "information types", which is equivalent to taking the Heyting algebra of the corresponding Kripke frame. The frame in [19] is different: "informations" are arbitrary non-empty sets of natural numbers. Then the intuitionistic logic is incomplete, and the corresponding set of valid formulas seems to be rather complex (recursively enumerable, but still unknown). Some related logics are studied in [28]; they also do not look simple. However the above proposition shows that in principle, Medvedev's idea is correct: completeness theorem holds if information is treated as a region of a certain kind (or a cone, in the relativistic approach).

## 10 Questions

As we have seen, in general, there are many natural relations between regions; only few of them have been studied from the modal logic viewpoint. So many questions in this field are open, and let us formulate some of them.

**1.** *Do there exist natural examples of decidable, but complex logics of regions?*

Note that now there is a big gap between undecidable logics (Proposition 24) and decidable logics, which are all in PSPACE.

**2.** *Do there exist natural examples of decidable logics of regions without the FMP?*

**3.** *(a) Do there exist "dimension axioms" in polymodal logics of regions? (b) Do "dimension axioms" exist for Minkowski spaces in the language with the"after" modality.*

**4.** *(a) Find modal properties of "light-accessibility" in Minkowski space.* $X \lambda Y := \mu(X - Y) = 0 \,\&\, t(Y) \geq t(X)$.

*(b) Find modal properties of the "inner contact" relation between balls.*

**5.** *(a) Do there exist decidable temporal logics of the forms* $\mathbf{L}(\mathbb{R}^n, \prec, \succ)$, $\mathbf{L}(\mathcal{B}_n, \subseteq, \supseteq)$ *etc. ?*

*(b) Do there exist finitely axiomatizable logics of this kind?*

Note that $\mathbf{L}(\mathbb{R}^n, \prec^{\pm})$ from Theorem 21 is the "omnitemporal fragment" of $\mathbf{L}(\mathbb{R}^n, \prec, \succ)$ corresponding to the modality "always" $\Box A := \Box_1 A \wedge \Box_2 A \wedge A$. Although Theorem 21 does not imply the non-finite axiomatazibility of this temporal logic, but may give some hints.

**6.** *Find properties of natural additional intuitionistic connectives in regional or relativistic logics.*

For example there is the "difference modality" $\blacktriangle$ with the following semantics:

u $\Vdash$ ▲$A$ iff $\exists$v(v $\Vdash$ $A$ & v $\asymp$ u);

so ▲$A$ is true if $A$ is true at some "distant place".

## 11 Acknowledgements

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