On modal logics of trees

Valentin Shehtman

October 21, 2020
Intuitionistic (propositional) formulas are in the standard language with a countable set of propositional letters $PL$ and the connectives $\lor, \land, \to, \bot$.

A superintuitionistic (propositional) logic is a set of intuitionistic formulas containing the standard intuitionistic axioms and closed under the rules

- (MP) $A, A \to B / A$;
- (Sub) $A / SA$, where $S$ is a propositional substitution.
Kripke semantics

An intuitionistic Kripke frame is a poset $F = (W, \leq)$.

A Kripke model over $F$ is a pair

- $M = (\Phi, \theta)$, where $\theta : PL \rightarrow 2^W$ is an intuitionistic valuation, i.e., for each $q$,
- $u \in \theta(q) \& u \leq v \Rightarrow v \in \theta(q)$.

The inductive definition of the truth of an intuitionistic formula $A$ at a point $u$ of a model $M$ ($M, u \models A$) is standard.

A formula $A$ is valid on a frame $F$ ($F \models A$) if $M, u \models A$ for every point $u$ of every model $M$ over $F$.

$\textbf{IL}(F) := \{A \mid F \models A\}$ is the superintuitionistic logic of a frame $F$.

$\textbf{IL}(C) := \bigcap\{\textbf{IL}(F) \mid F \in C\}$ is the superintuitionistic logic of a class of frames $C$, or the superintuitionistic logic determined by $C$. 
Logics of the form $\mathbf{IL}(C)$ (or, equivalently, $\mathbf{IL}(F)$) are called (Kripke) complete.

Logics of the form $\mathbf{IL}(C)$, where $C$ is a class of finite frames, are said to have the finite model property (FMP).

Every finitely axiomatizable logic with the FMP is decidable.
DEFINITION (by J. Drugush)
A tree is a poset \((W, \leq)\) with the following properties

- for any \(u\), the set \(\{v \mid v \leq u\}\) is a chain,
- \((W, \leq)\) is a lower semilattice.

REMARK. This is a rather broad definition. In particular, every chain is a tree in this sense.

A forest is a disjoint union of trees. A forest logic is a logic determined by a forest.

Theorem (Drugush, 1984) Every superintuitionistic forest logic has the FMP, moreover, it is determined by a forest consisting of finite trees.
Modal formulas are build from the set $PL$ of proposition letters using the connectives $\rightarrow, \bot, \square$. Other connectives ($\land, \lor, \diamond, \top$ etc.) are abbreviations.

The modal depth $d(A)$ of a modal formula $A$ is defined by induction.

- $d(q) = 0$ for $q \in PL$, $d(\bot) = 0$,
- $d(A \rightarrow B) = \max(d(A), d(B))$,
- $d(\square A) = d(A) + 1$.

A modal logic is a set of modal formulas containing

- the classical tautologies;
- the axiom of $\textbf{K}$: $\square(p_1 \rightarrow p_2) \rightarrow (\square p_1 \rightarrow \square p_2)$,

and closed under the rules

- (MP) $A, A \rightarrow B / A$;
- (Nec) $A / \square A$;
- (Sub) $A / SA$, where $S$ is a propositional substitution.

The minimal modal logic is $\textbf{K}$. 
An **Kripke frame** is a non-empty set with a binary relation $F = (W, R)$.

A **Kripke model over $F$** is a pair $M = (\Phi, \theta)$, where $\theta : PL \rightarrow 2^W$ is a valuation.

The inductive definition of the truth of a modal formula $A$ at a point $u$ of a model $M$ ($M, u \models A$) is standard.

A formula $A$ is **valid** on a frame $F$ ($F \models A$) if $M, u \models A$ for every point $u$ of every model $M$ over $F$.

$L(F) := \{ A \mid F \models A \}$ is the **modal logic** of a frame $F$.

$L(C) := \bigcap \{ L(F) \mid F \in C \}$ is the **modal logic** of a class of frames $C$, or the **modal logic determined by** $C$. 

Logics of the form $L(C)$ (or, equivalently, $L(F)$) are called (Kripke) complete.

Logics of the form $L(C)$, where $C$ is a class of finite frames, are said to have the finite model property (FMP).

**FACT 1** Every finitely axiomatizable logic with the FMP is decidable.

A modal logic $L$ is

- **locally tabular** if for any $n$ there exists finitely many formulas in proposition letters $p_1, \ldots, p_n$, up to equivalence in $L$ ($L \vdash A \leftrightarrow B$).
- **tabular** if it is determined by a single finite frame.

**FACT 2** Every tabular logic is locally tabular.

**FACT 3** Every locally tabular logic has the FMP.

**FACT 4** Every extension of a locally tabular logic is locally tabular.
Definition

For a set of modal formulas $\Gamma$, put

$$\Box \Gamma := \{ \Box A \mid A \in \Gamma \}.$$  

If $L = K + \Gamma$ for a set of formulas $\Gamma$, put

$$\Box \cdot L := K + \Box \Gamma.$$  

Lemma

$K + \Gamma \vdash A$ implies $K + \Box \Gamma \vdash \Box A$.

So $K + \Gamma = K + \Delta$ implies $K + \Box \Gamma = K + \Box \Delta$, i.e., $\Box \cdot L$ is well-defined. It turns out that $\Box \cdot L$ inherits many properties of $L$. 
Theorem 1

- If \( L \) is Kripke complete, then \( \Box \cdot L \) is Kripke complete.
- If \( L \) has the FMP, then \( \Box \cdot L \) has the FMP.
- If \( L \) is locally tabular, then \( \Box \cdot L \) is locally tabular.

Since the logic \( \text{Triv} := K + (p \leftrightarrow \Box p) \) is tabular (it is determined by a single reflexive point), we obtain many examples of locally tabular logics:

Corollary

*The logics \( K + \Box^n (p \leftrightarrow \Box p) \) (and all their extensions) are locally tabular.*
Trees

Definition

A tree is a rooted frame, in which every point, but the root, has a single predecessor. I.e., this is a frame \((W, R)\) with a point \(u\) such that

- \(W = \bigcup_{n \geq 0} R^n(u)\),
- \(\forall x \neq u \exists! y \ yRx\).

A reflexive tree is a reflexive closure of a tree. Similarly we define transitive trees, symmetric trees, etc.

This can be done for any first-order condition on frames expressed by a Horn sentence.
Theorem 2
A modal logic determined by any class of reflexive trees has the FMP.

Proof
(Sketch.) It suffices to consider $L(F)$ for a single reflexive tree $F$. Suppose $F, x \not\models A$ for some point $x$. Let $G = F \uparrow x$ be the subtree of $F$ starting at $x$, and let $n = d(A)$. Consider its truncation $G^{(n)} = G \upharpoonright R^n(x)$. Then $G^{(n)}, x \not\models A$; this is proved for example, by playing a bisimulation game between a countermodel $M$ for $A$ in $G$ and the truncated model $M^{(n)}$, so $M, x$ and $M^{(n)}, x$ are $n$-bisimilar. Thus every formula refuted in $F$ is refuted in some $G^{(n)}$.
On the other hand, there is a p-morphism $f : G \rightarrow G^{(n)}$ such that
• $f(u) = u$ for $u \in G^{(n)}$,
• $f(u) = v$ for $xR^nvR^mu$ (there are unique such $v$ and $m$).
Reflexive trees (continued)

Proof

(Continued) Thus

\[
L(F) \subseteq L(G) \subseteq L(G^{(n)}).
\]

It follows that

\[
L(F) = L(\{(F \uparrow x)^{(n)} \mid x \in F, n \geq 1\}).
\]

Finally observe that every logic \( L((F \uparrow x)^{(n)}) \) is locally tabular. This follows from a Theorem in


It states that every logic axiomatized by Chagrov’s formula (forbidding paths of different points of length \( > n \)) is locally tabular. Therefore, \( L(F) \) has the FMP.
Serial trees

A serial frame is a frame without endpoints (where $R(x) = \emptyset$). Equivalently, $F$ is serial iff $F \models \Diamond \top$.

**Theorem 3**

A modal logic determined by any class of serial trees has the FMP.

**Proof**

(Sketch.) Again it suffices to consider $\mathbf{L}(F)$ for a single serial tree $F$. The method is almost the same as in Theorem 2. Now we take a truncation $G^{(n)}$ and make all its endpoints reflexive. This gives us a serial frame $G^{(n)}\bullet$, and still $G^{(n)}\bullet, x \not\models A$.

Since $G$ is serial, we have the same p-morphism $f : G \rightarrow G^{(n)}\bullet$.

Finally note that $G^{(n)}\bullet \models \square^n (p \leftrightarrow \square p)$, so by Theorem 1, $\mathbf{L}(G^{(n)}\bullet)$ is locally tabular, and we can apply the same argument as in Theorem 2.
More examples of FMP

Theorem 4

The logic of every class of trees validating

$$\Diamond \top \rightarrow \Diamond^2 \top \land \Diamond \Box \bot$$

has the FMP.

Theorem 5

A modal logic determined by any class of reflexive symmetric trees has the FMP.
Counterexample

Theorem 6

There exist a countable tree $F$ such that $L(F)$ lacks the FMP.

Proof

(Sketch, joint with A. Alexeev). Consider the formula

$Alt_1 := \Diamond p \rightarrow \Box p$. It is well-known that $(W, R) \models Alt_1$ iff every $x$ has at most one successor.

Let

$L_0 := \mathbf{K} + \Box Alt_1 + \{Alt_1 \lor (\Diamond^n \Box \bot \rightarrow \Diamond^{n+1} \Box \bot) \mid n \geq 1\}.$

Then

- Every finite frame validating $L_0$ validates $Alt_1 \lor \Box \Diamond \top$.
- $L_0 \not\models Alt_1 \lor \Box \Diamond \top$: there is a corresponding infinite tree $F$.

So $L_0$, as well as $L(F)$, lacks the FMP.
Some questions

1. Does there exist a transitive tree $F$ validating $\text{GL}$ such that $\text{L}(F)$ lacks the FMP? (Conjecture: yes).
2. Does the modal logic of every reflexive transitive tree enjoy the FMP?
3. Do there exist continuum many superintuitionistic forest logics?