Computational aspects in Multidimensional Symbolic Dynamics

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Plan

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1D SFT

2D SFT

Beyond

Conclusion

1D SFT

- **alphabet**: finite set $\mathcal{A}$ (e.g. $= \{\square, \blacksquare\}$)
- **configuration**: $x \in \mathcal{A}^{\mathbb{Z}}$
- $u \in \mathcal{A}^*$ **appears** in x; $u \sqsubset x$: $u = x_{i \cdots i+|u|-1}$ for some $i \in \mathbb{Z}$
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subshift $\Leftrightarrow$ shift-invariant+closed (product topology)
codings of dynamical systems w.r.t finite partitions

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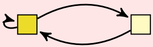

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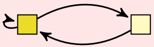

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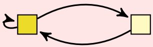

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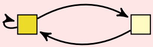

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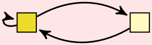

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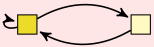

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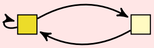

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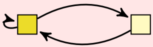

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- **sofic**: image of SFT by alphabet projection $\pi : \mathcal{A} \to \mathcal{B}$
- **language** of $\mathcal{X}$: $\mathcal{L}(\mathcal{X}) := \{ u \sqsubset x | x \in \mathcal{X} \}$
  $\mathcal{L}(\mathcal{Y})$ is regular $\Leftrightarrow$ Y is sofic

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2D SFT

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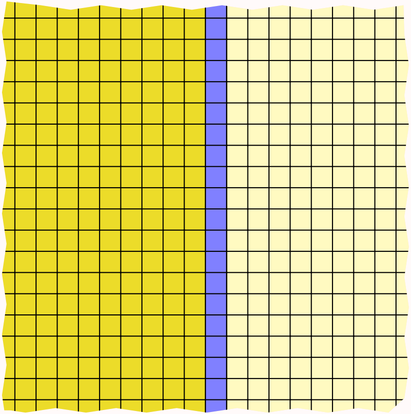

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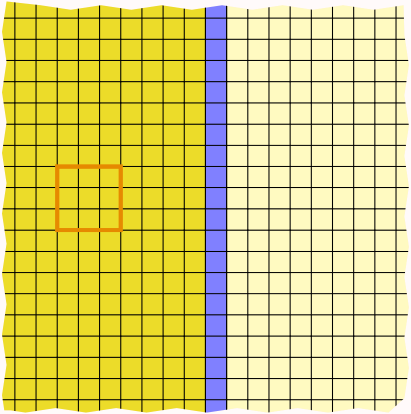

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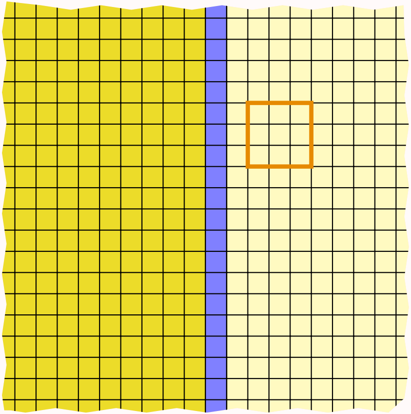

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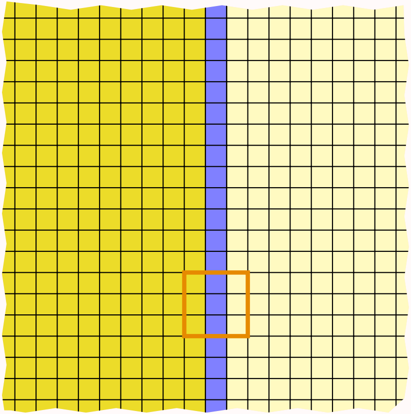

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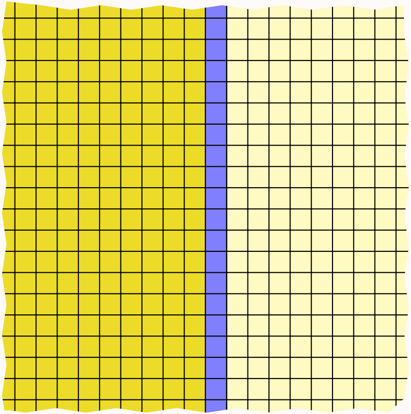

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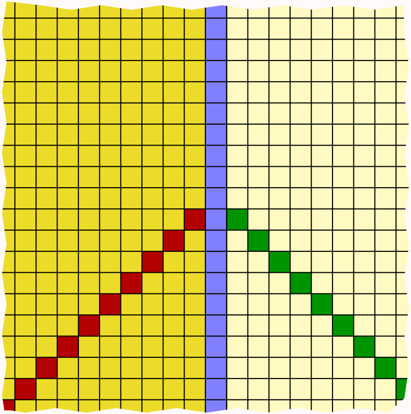

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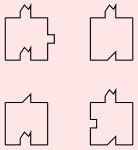

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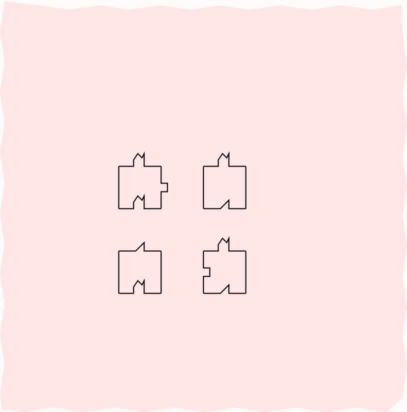
- x is **periodic** if $\mathfrak{P}(x) \sim \mathbb{Z}^2$.



SFT $\leftrightarrow$ Wang tile sets

Cellular automata space-time

- **cellular automaton**

  some **local rule** $f : \mathcal{A}^{2r+1} \to \mathcal{A}$
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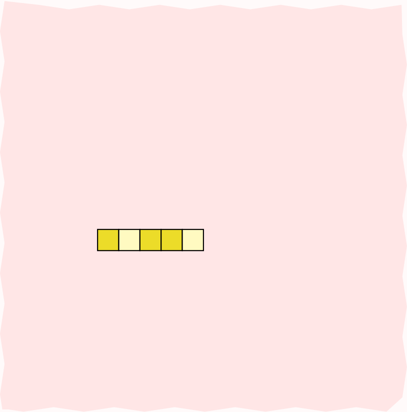

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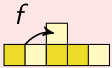

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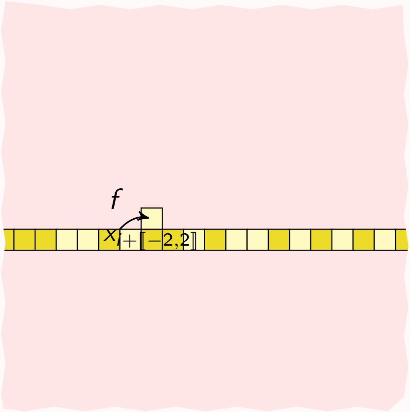

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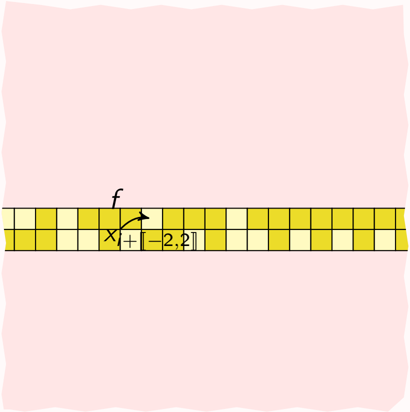

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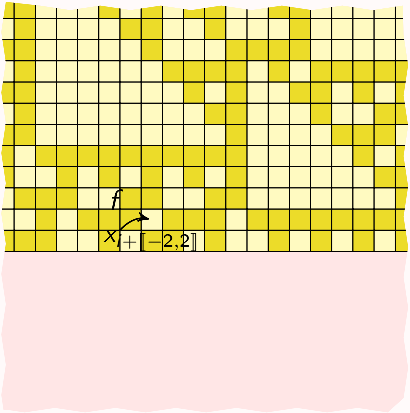

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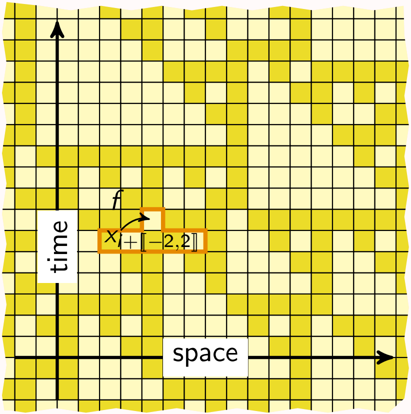

F cellular automaton
$\leftrightarrow$ F continous and

$$\begin{array}{ccc} \mathcal{A}^{\mathbb{Z}} & \xrightarrow{\sigma} & \mathcal{A}^{\mathbb{Z}} \\ \downarrow F & & \downarrow F \\ \mathcal{A}^{\mathbb{Z}} & \xrightarrow{\sigma} & \mathcal{A}^{\mathbb{Z}} \end{array}$$

where $\sigma(x)_i = x_{i+1}$ is the **shift**.

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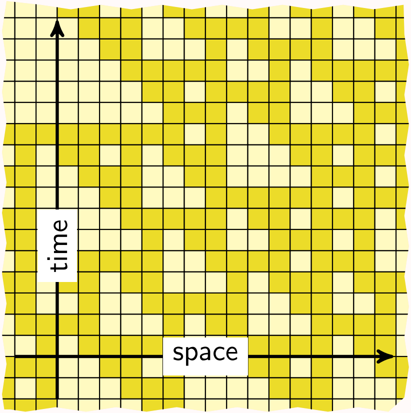

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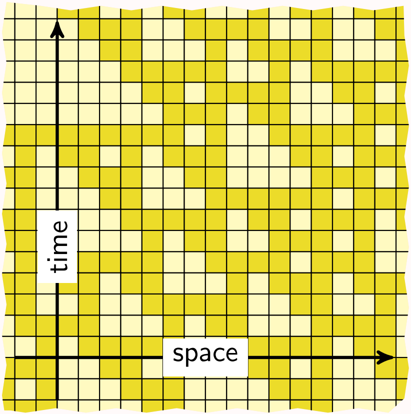

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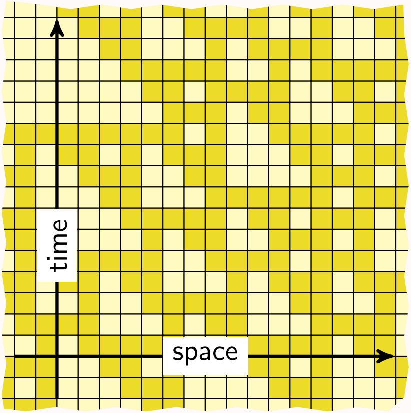

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- universal TM
  $\rightarrow$ SFT with undecidable language

Trying to build configurations

Given a finite list $\mathcal{F}$ of forbidden patterns:

- $\exists$ semi-algo trying to tile all squares $n \times n$.
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- $\exists$ algo answering $\mathcal{X}_{\mathcal{F}} = \emptyset$?
  $\checkmark$if $\nexists \mathcal{X}_{\mathcal{F}}$ **aperiodic** (=$\mathcal{X}_{\mathcal{F}} \neq \emptyset$ and no $x \in \mathcal{X}_{\mathcal{F}}$ is periodic)

Aperiodic SFT

- from substitutions:
  [Berger '64], [Robinson '71],
  [Mozes '88], [Ollinger '08]
- from restriction of 3 periodic layers:
  [Aandera-Lewis '74]
- from piecewise-affine map:
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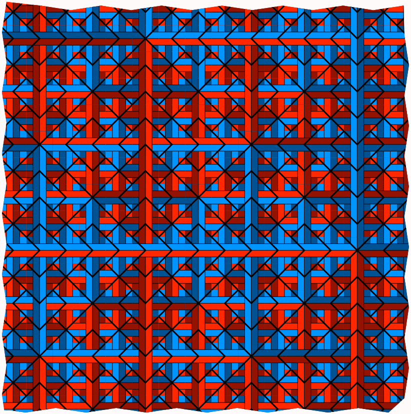

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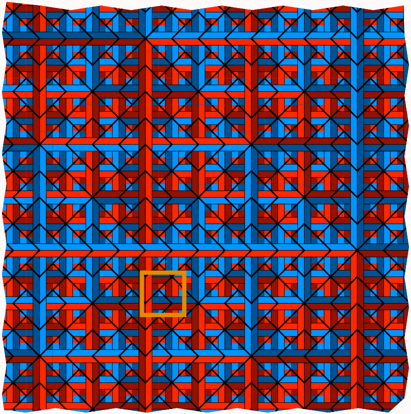

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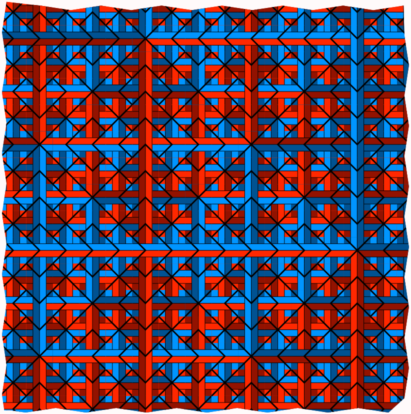

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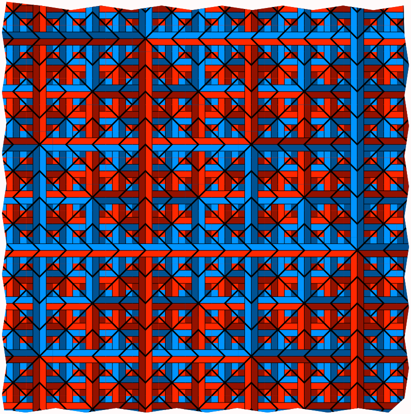

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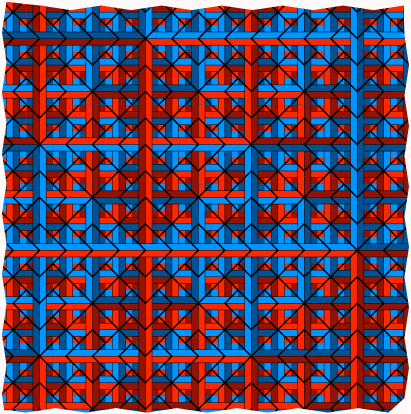
- computational model.
  What can SFT compute?



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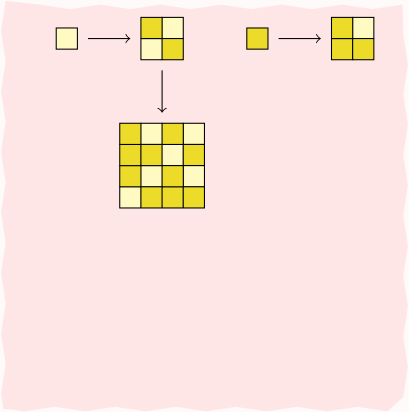

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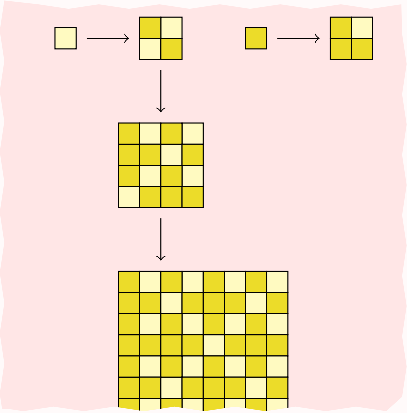
$$\mathcal{X}_\tau := \left\{ x \in \mathcal{A}^{\mathbb{Z}^2} \,\middle|\, \begin{array}{r} \forall u \sqsubset x, \exists k, a, \\ u \sqsubset \tau^k(a) \end{array} \right\}$$



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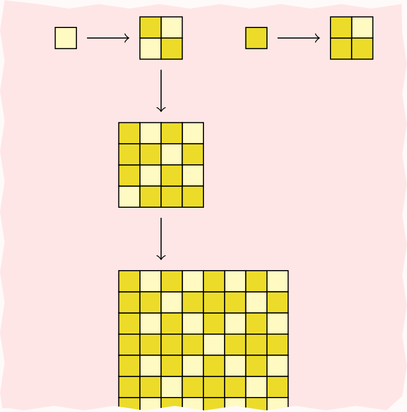
Thm [Mozes '88]:
$\mathcal{X}_\tau$ is sofic.



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$\tau_1 \tau_2 \ldots \in \mathcal{S}^{\mathbb{N}}$, where $|\mathcal{S}| < \infty$.
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Thm [Aubrun-Sablik '14]:
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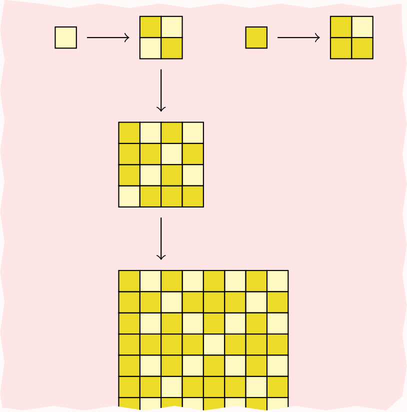

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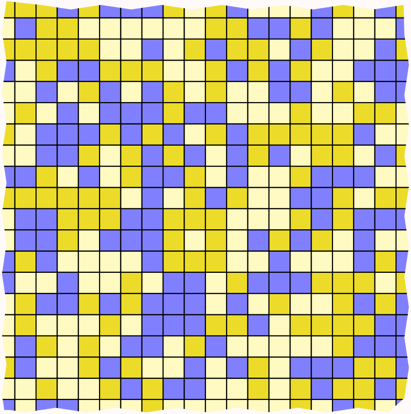

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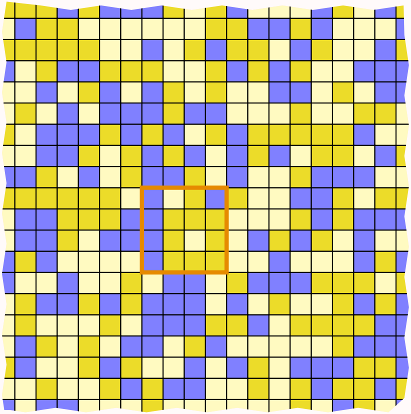

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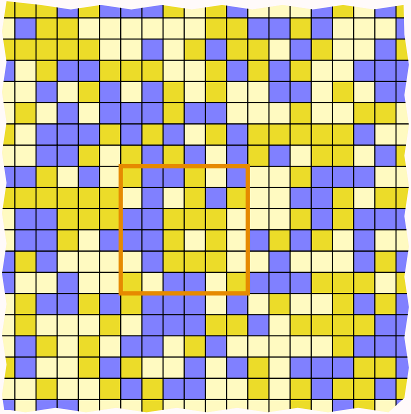

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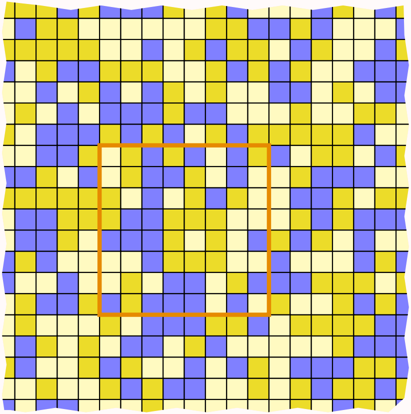

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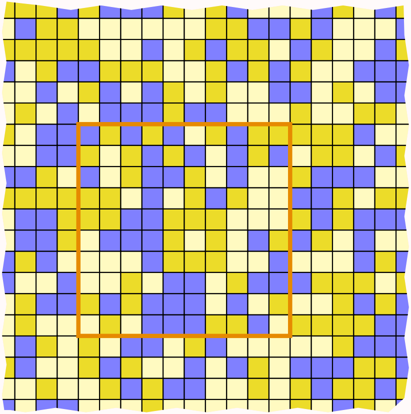

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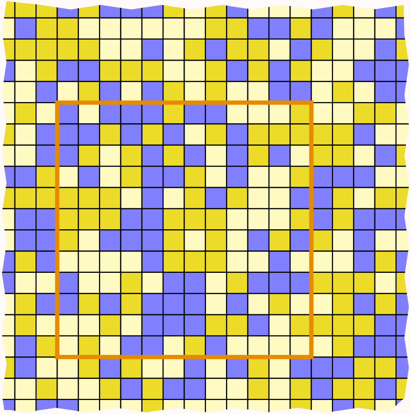

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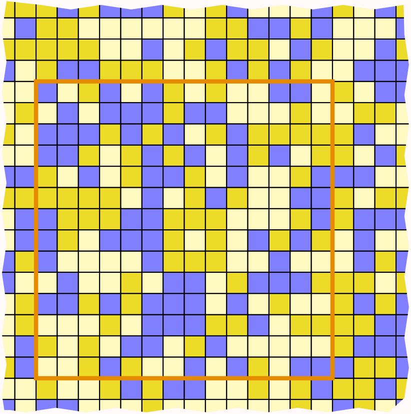

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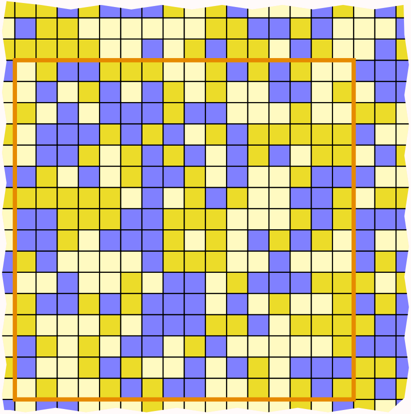

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Π_1^0: limit of nonincreasing computable $(h_n) \in \mathbb{Q}^{\mathbb{N}}$.

Thm:
The entropy of an SFT
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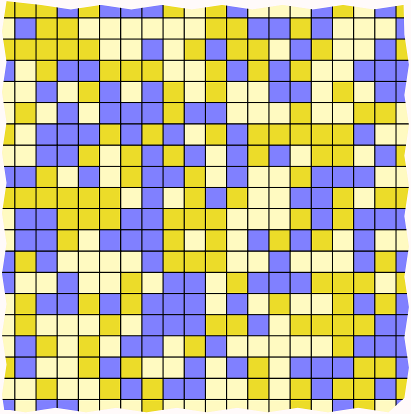

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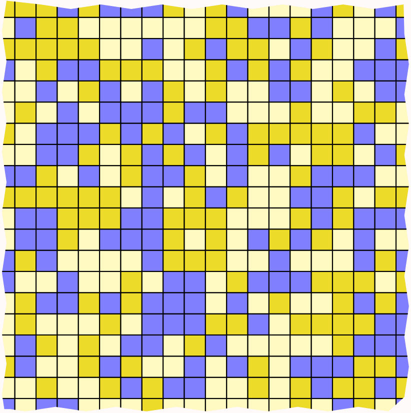
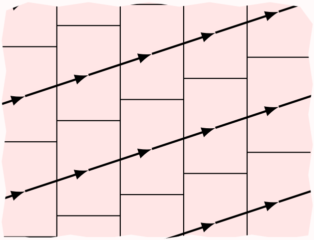
Thm [Hochman-Meyerovitch '08]:
The entropy of an SFT can be any λ in $\Pi_1^0 \cap \mathbb{R}_+$.



Periods

Thm [Gurevich-Koryakov '72]: $\exists x$ periodic? undecidable.



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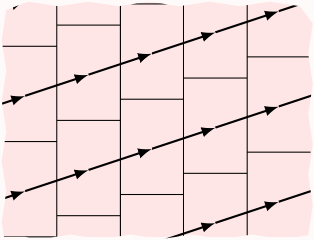
$\left\{ n \middle| \exists x \in \mathcal{X}, \mathfrak{P}(x) = n\mathbb{Z}^2 \right\}$ is* in NP.

* Subsets of $\mathbb{N}$ are represented in unary (as $\subset 1^*0^*$).



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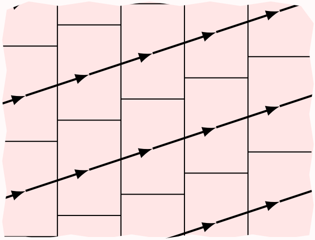
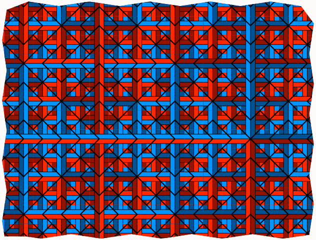


Uniform recurrence

x is **uniformly recurrent** if:
$\forall u \sqsubset x, \exists q_x(u), \forall \mathbf{i} \in \mathbb{Z}^2, u \sqsubset x_{\mathbf{i}+[\![0,q_x(u)]\!]^2}$.

Thm [Birkhoff 1912]: $\forall \mathcal{X}$ subshift, $\exists x \in \mathcal{X}$ uniformly recurrent.



Uniform recurrence

x is **uniformly recurrent** if:
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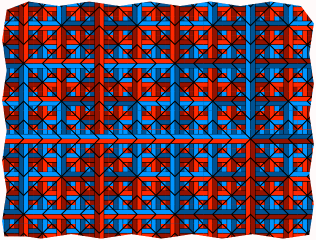
Thm [Ballier-Jeandel '10]: $\exists \mathcal{X}$ SFT $\neq \emptyset$, $\forall x$ uniformly recurrent, q_x is not computably bounded.



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automorphism of $\mathcal{X} \subset \mathcal{A}^{\mathbb{Z}^2}$: $F : \mathcal{X} \to \mathcal{X}$ shift-commuting homeo.
$\leftrightarrow$ reversible cellular automata [Curtis-Hedlund-Lyndon '69]

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Let $\mathcal{X} \subset \mathcal{A}^{\mathbb{Z}^2}$ an SFT, and $F_1, \ldots, F_k$ automorphims.
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Then the word problem in $\langle F_1, \ldots, F_k \rangle$
is as complex* as any c.e. problem.

* In terms of Turing degrees, or enumeration degrees;
in particular it can be uncomputable.

Computability-theoretical invariants

Thm [Myers '74, Hanf '74]: There exists an acomputable 2D SFT ($\neq \emptyset$ + no computable configuration).

Thm [Simpson '11]:
Every Medvedev (and every Muchnik) degree contains a 2D SFT.

Thm [Jeandel-Vanier '11]: For every set of Turing degrees of a Π^0_1 set, there exists an SFT with the same degrees + 0.

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- **effective subshift**: $\mathcal{X}_{\mathcal{F}}$ for some c.e. $\mathcal{F}$

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A parenthesis on effective closedness

product topology in $\mathcal{A}^{\mathbb{N}}$:

- **closed** set: forbid a set of prefixes
- **clopen** set: forbid a finite set of prefixes
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symbolic dynamics:

- **subshift**: forbid a set of factors
- **SFT**: forbid a finite set of factors
- **effective subshift**: forbid a c.e. set of factors

Expansive directions

A stripe L is **expansive** for $\mathcal{X}$ if $\forall x \neq y \in \mathcal{X}, x_{|L} \neq y_{|L}$.
Its slope $\theta \in \mathbb{R} \bmod \mathbb{Z}$ is an **expansive direction**.

Thm :
The set of expansive directions is effectively open.

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$\rightarrow$ undecidability of group automata finiteness, order...
[Gillibert '13, Gillibert '17]

Restrictions of 2D SFT

- computation with deterministic directions
  ($\sim$ CA space-time diagrams) [G.-Zinoviadis '15]
- computation robust to errors
  [Durand-Romashchenko-Shen '10]
- computation within uniformly recurrent SFT
  [Durand-Romashchenko '17]
- aperiodic Wang 11-tile sets
  [Jeandel-Rao '15]
- quantify aperiodiciy: dimension group, cylindricity, gender...

Plan

1D SFT

2D SFT

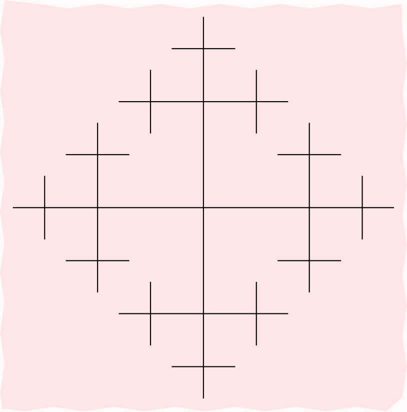
Beyond

Conclusion

Other groups

Subshifts in A^G; G a f.g. group.

- Is there an aperiodic SFT?
- Is SFT emptiness undecidable?
- How powerful are SFT?



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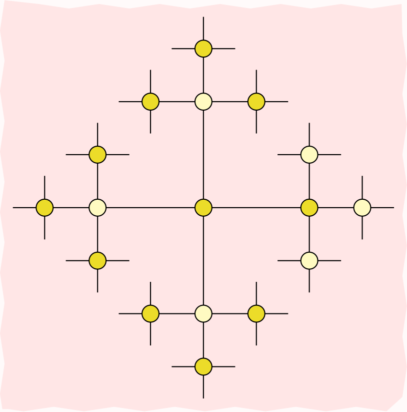

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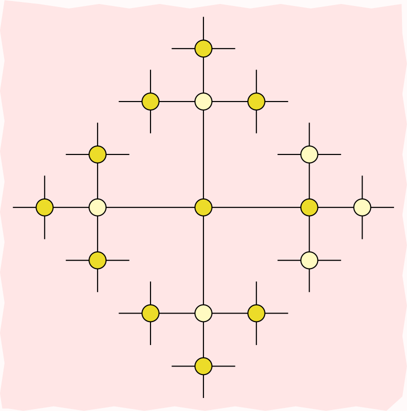

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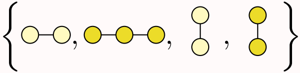

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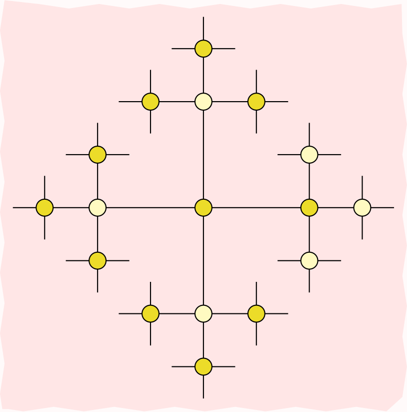
Equivalence:

- virtually free group;
- Cayley graph with finite tree-width;
- context-free word problem;
- decidable MSO logic;
- decidable emptiness problem?



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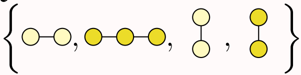


Plan

Conclusion

Known:

- 1D SFT: power of finite automata.
- 2D SFT: power of Turing machines.
- undecidable dynamics.
- computational results over SFT $\sim$ over f.g. groups: closure spaces [Jeandel '15].

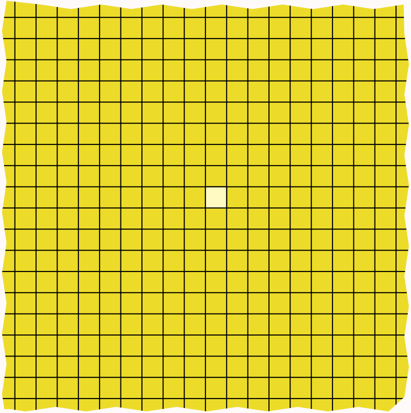
To be known:

- Relevant sufficient conditions (on Kolmogorov complexity?) for soficity of 2D effective subshifts?
- Quantifying aperiodicity?
- Complexity theory in SFT?
- Which groups admit undecidable SFT emptiness problem / aperiodic SFT?

2D sofic subshifts

Examples of sofic subshifts on $\mathcal{A} = \{\square, \blacksquare\}$.

- $\mathcal{Y} = \{x \mid \sharp\square \leq 1\}$



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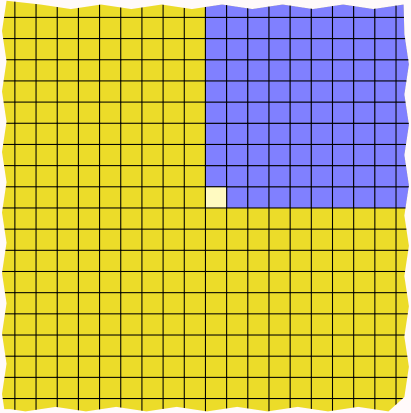

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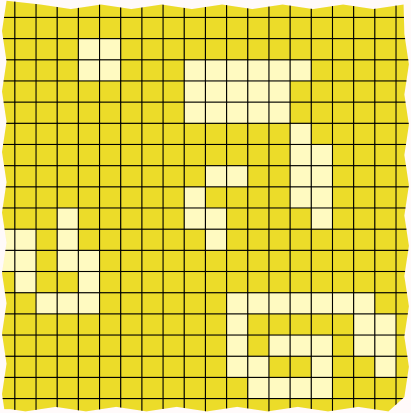

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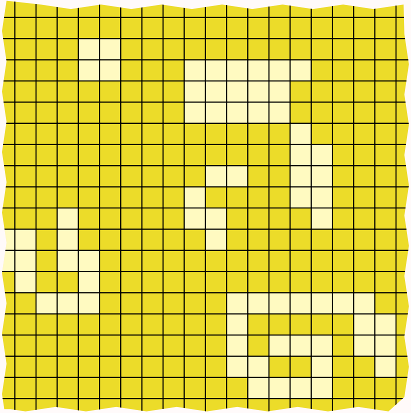
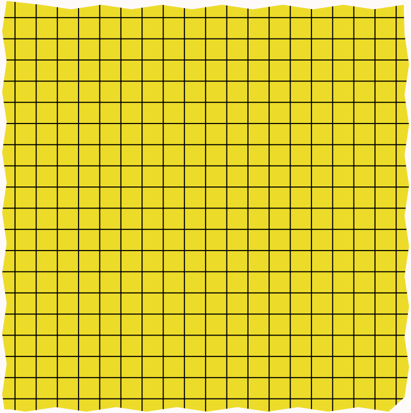


Necessary conditions for soficity

Any sofic subshift $\mathcal{Y}$ has:

- co-c.e. language $\mathcal{L}(\mathcal{Y})$



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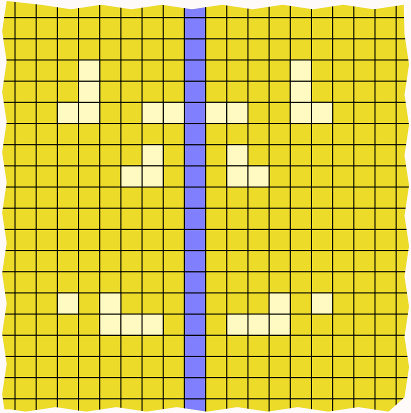

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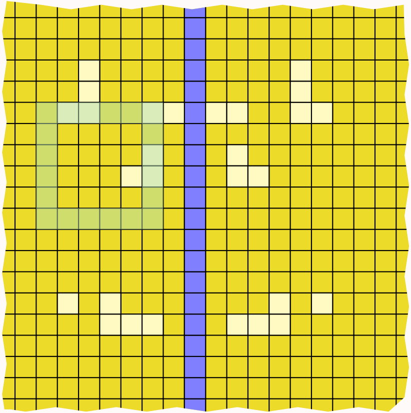

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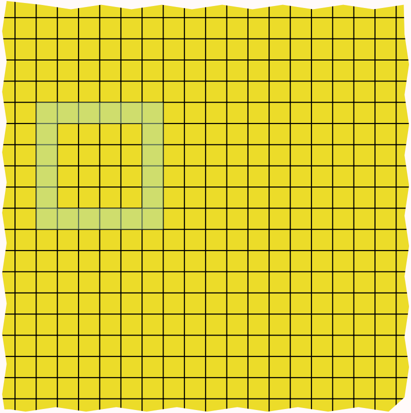

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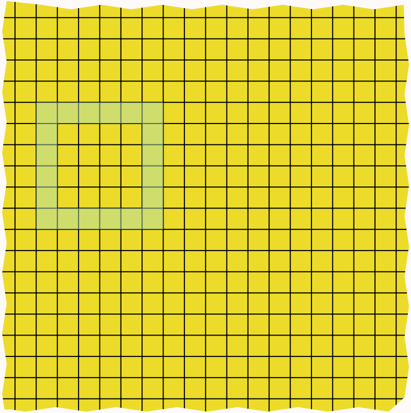

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