The modal logic of almost sure frame validities in the finite

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The story of 0-1 law for modal frame validity in the finite: a small scientific drama in 3 parts + epilogue

I. J. Halpern and B. Kapron,
Zero-one laws for modal logic,
Annals of Pure and Applied Logic, 1994.

II. V. Goranko and B. Kapron,
The modal logic of the countable random frame, 2000.
Archive for Mathematical Logic, 2003.

III. J.-M. Le Bars,
Zero-one law fails for frame satisfiability in propositional modal logic,
Proceedings of LICS'2002.

VI. J. Halpern and B. Kapron,
Erratum to "Zero-one laws for modal logic"
Ann. Pure Appl. Logic, 2003.

Outline

I. Brief background on asymptotic probabilities of logical formulae, 0-1 laws, and almost sure validities.

II. Some modal logic background.

III. On the modal logic of the countable random frame
and the modal logic of almost sure frame validities in the finite.

Asymptotic probabilities of logical formulae, 0-1 laws, and almost sure validities

Finite random structures and asymptotic probabilities

Consider a fixed finite relational signature **L**. (E.g., a single binary relation.)

- Define probability space S_n on the set of n-element **L**-structures over the set $\{1, \ldots, n\}$ (assume uniform distribution).
- Define classical discrete probability of a property $\mathfrak{P}$ on the (finite) space S_n: $\mathrm{Pr}_n(\mathfrak{P})$. This is the labelled probability of the property $\mathfrak{P}$.
- Define asymptotic probability of a property $\mathfrak{P}$:

  $\mathrm{Pr}(\mathfrak{P}) = \lim_{n\to\infty} \mathrm{Pr}_n(\mathfrak{P})$

  if this limit exists.

Remark: This probability measure is not countably additive.

If $\mathrm{Pr}(\mathfrak{P}) = 1$, $\mathfrak{P}$ is said to be almost surely true (valid) in the finite.

If $\mathrm{Pr}(\mathfrak{P}) = 0$, $\mathfrak{P}$ is almost surely false (invalid) in the finite.

Almost surely true and false properties of (di)graphs

A binary relational structure $F = \langle W, R \rangle$ will hereafter be called a frame.

(Thus, a digraph is a loopless frame; a graph is a digraph with a symmetric edge relation.)

Some almost surely true properties on graphs:

connected, containing a k-clique, Hamiltonian, etc.

Also: rigid (no non-trivial automorphisms).

Consequently, for all abstract properties $\mathfrak{P}$ the probabilities over labelled and unlabelled structures (i.e. up to isomorphism) are equal.

Some almost surely false properties:

- On graphs: planar, k-colourable, Eulerian, tree-like, etc.
- On digraphs: the edge relation to be symmetric, transitive, ...
  ... every non-trivial universal property. Also, e.g. rootedness.

(In particular, a randomly chosen frame is almost surely not a digraph. Furthermore, a randomly chosen digraph is almost surely not a graph.)

Countable random structures

Fix any finite relational language **L**.
There is a special 'countable random structure' $\mathcal{R}_{\mathbf{L}}$ for **L**
(Erdös-Rényi model; for graphs: Rado graph), such that any uniformly randomly chosen countable **L**-structure is isomorphic with probability 1 to $\mathcal{R}_{\mathbf{L}}$.

There is a simple logical characterization of $\mathcal{R}_{\mathbf{L}}$, by means of an infinite set of extension axioms.

Extension axioms for the countable random frame

For every $n \in \mathbb{N}$, the extension axiom scheme $(\mathsf{EXT})_n^F$ for frames is as follows:

$$(\mathsf{EXT})_n^F = \forall \overline{x} \exists y \left(\bigwedge_{i \neq j} x_i \neq x_j \to \left(\bigwedge_{i \in U_n} x_i \neq y \wedge T(y) \wedge \right.\right.$$

$$\left.\left. \bigwedge_{i \in I} Rx_i y \wedge \bigwedge_{i \in U_n \setminus I} \neg Rx_i y \wedge \bigwedge_{i \in J} Ryx_j \wedge \bigwedge_{j \in U_n \setminus J} \neg Ryx_j \right)\right)$$

where $\overline{x} = x_1, \ldots, x_n$ and $T(y)$ is either Ryy or $\neg Ryy$.

$(\mathsf{EXT})_n^F$ says that for every n different points in the frame there is a point which is related to and from each of those, and with itself, in any consistent explicitly prescribed way. (Saturation.)

Gaifman's theorem

Theorem

(Gaifman'64) A countable **L***-structure satisfies all extension axioms iff it is isomorphic to* $\mathcal{R}_{\mathbf{L}}$.

Generalizes Cantor's theorem characterizing the order of the rationals.

Corollary

The FO theory axiomatized with all extension axioms is ω*-categorical and has no finite models, hence it is complete and decidable.*

The unique countable model of the extension axioms for frames:
the countable random frame F^{r}.

Fagin's theorem and 0-1 laws

Lemma. (Fagin'75) *Every extension axiom is almost surely true in the finite.*

Proof: Combinatorial estimation of the probabilities of the extension axioms.

Theorem (Transfer theorem, Fagin'1975)

For every **L***-sentence* ψ *the following are equivalent:*

1. $\mathcal{R}_{\mathbf{L}} \models \psi$.
2. ψ *follows from (finitely many) extension axioms.*
3. ψ *is almost surely true in the finite.*

Corollary (0-1 law for first-order logic (FO))

Every first-order property of relational structures is either almost surely true or almost surely false in the finite.

This result was first established in 1969 in:
Глебский Ю. В., Коган Д. И., Лиогонький М. И., Таланов В. А. Объём и доля выполнимости формул узкого исчисления предикатов, in: Кибернетика.
The proof idea was completely different: 'almost sure' quantifier elimination.

The modal logic of almost sure validity in Kripke models

Two notions of validity in a structure of modal formulae:
in a Kripke model (model validity), and in a Kripke frame (frame validity).

Model validity is a *first-order* property. Therefore:

Proposition (Corollary from Fagin's theorem)

The transfer theorem and the zero-one law hold for validity of modal formulae in Kripke models.

Proposition (Halpern & Kapron, APAL'94)

*The modal logic of almost sure model validity is Carnap's modal logic **C**, axiomatized by the set of axioms:*

$$\{\Diamond\phi \mid \phi \textit{ is a consistent classical propositional formula}\}$$

NB: Carnap's logic is not closed under uniform substitution.

0-1 law for frame validity in modal logic?

Is it true that every modal formula is either almost surely valid or almost surely invalid in finite frames?

Frame validity of a modal formula is not a FO, so 0-1 law for it does not follow from Fagin's theorem.

Indeed, the analogue of the transfer theorem for modal frame validity was refuted in (G. & Kapron, 2003).

The 0-1 law, claimed to hold for modal frame validity in (Halpern & Kapron '94) was later refuted in (Le Bars, LICS'2002).

The theory of almost sure validities in the finite for a given logic

For a logic **L** with semantics defined, inter alia, on arbitrarily large finite models, let $\mathbf{L}^{as}$ be the set of formulae of **L** that are almost surely true in the finite.

$\mathbf{L}^{as}$ contains all validities of the logic **L** and is closed under finitary logical consequence (because of the finite additivity of Pr).

So, $\mathbf{L}^{as}$ is a well defined *logical theory (logic) over **L***.

A *generic problem*:
Find a purely logical (axiomatic or model-theoretic) characterization of L^{as}.

The 'easy' case:
If 0-1 law holds for **L** due to transfer from truth in $\mathcal{R}_{\mathbf{L}}$, then $\mathbf{L}^{as} = Th(\mathcal{R}_{\mathbf{L}})$.

What if there is no such structure $\mathcal{R}_{\mathbf{L}}$, or no transfer theorem, or 0-1 law fails?

Then $\mathbf{L}^{as}$ still exists, but it may be very hard to characterise logically.

This seems to be the case with modal logic and frame validity.

Some modal logic background

Bounded morphisms and kernel partitions

Let $\mathsf{F}_1 = \langle W_1, R_1 \rangle$ and $\mathsf{F}_2 = \langle W_2, R_2 \rangle$ be frames. A mapping $h : W_1 \to W_2$ is a bounded morphism (aka, p-morphism) from F_1 to F_2 if the following hold:

(i) For all $x, y \in W_1$, if xR_1y then $h(x)R_2h(y)$.

(ii) For all $x \in W_1, t \in W_2$, if $h(x)R_2t$ then xR_1y for some $y \in W_1$ such that $h(y) = t$.

If h is onto, F_2 is called a bounded-morphic image of F_1.

$\mathsf{F}_1 \twoheadrightarrow \mathsf{F}_2$ will denote the claim that F_2 is bounded-morphic image of F_1.

Recall that frame validity of modal formulae is preserved under bounded morphisms i.e., if $\mathsf{F}_1 \models \phi$ and $\mathsf{F}_1 \twoheadrightarrow \mathsf{F}_2$, then $\mathsf{F}_2 \models \phi$.

Every bounded morphism $h : \mathsf{G} \to \mathsf{F}$ determines a kernel partition $\mathcal{P}_{\mathsf{F}}$ in G, consisting of the family of clusters $\{h^{-1}(w) \mid w \in W_{\mathsf{F}}\}$.

Conversely, for every kernel partition in G generated by mapping $h : \mathsf{G} \to \mathsf{F}$ and satisfying certain conditions, the mapping h is a bounded morphism from G onto F.

Thus, there is a 1-1 correspondence between the kernel partitions satisfying these conditions and the bounded morphic images of any given frame.

The countable random frame F^{r} and frames of diameter 2

Recall: the countable random frame F^{r} is the unique countable model of the extension axioms for frames.

An important fact about F^{r} from (G. & Kapron, 2003):

- F^{r} has a diameter 2: every point can be reached from any point (incl. itself) in 2 steps. Indeed, by $(\mathsf{EXT})_2$: $\mathsf{F}^{\mathsf{r}} \models \forall x_1 \forall x_2 \exists y (R x_1 y \wedge R y x_2)$.

Since $(\mathsf{EXT})_2$ is almost surely true in the finite, the subset $\mathcal{F}^{\mathsf{d2}}$ of all finite frames of diameter 2 has asymptotic measure 1 in the set $\mathcal{F}^{\mathsf{fin}}$ of all finite frames.

Consequently, every property of frames is almost surely true in $\mathcal{F}^{\mathsf{fin}}$ iff it is almost surely true in $\mathcal{F}^{\mathsf{d2}}$.

The universal modality is almost surely definable in ML

ML_{U} – the basic modal language extended with *universal modality* $[\mathsf{U}]$ (and *existential modality* $\langle \mathsf{U} \rangle$).

$[\mathsf{U}]$ and $\langle \mathsf{U} \rangle$ are simply definable in every frame in $\mathcal{F}^{\mathsf{d2}}$:

$$[\mathsf{U}]p \equiv \Box\Box p, \quad \text{respectively} \quad \langle \mathsf{U} \rangle p \equiv \Diamond\Diamond p.$$

Therefore, these equivalences hold in almost every finite frame, and also in F^{r}.

Hereafter we treat $[\mathsf{U}]$ and $\langle \mathsf{U} \rangle$ either as primitives or as definable as above. All results presented further are respectively the same in both cases.

Characteristic formulae (a lá Jankov-Fine)

Let $\mathsf{F} = \langle W, R \rangle$ be any finite frame with $W = \{w_1, \ldots, w_n\}$ and let $\{p_1, \ldots, p_n\}$ be fixed different propositional variables.

The characteristic formula of F over $\langle p_1, \ldots, p_n \rangle$ is the ML_{U}-formula

$$\chi_{\mathsf{F}}(p_1, \ldots, p_n) := \neg[\mathsf{U}]\, \delta_{\mathsf{F}}(p_1, \ldots, p_n),$$

where δ_{F} is the 'modal diagram' of F, with $p_1, \ldots, p_n$ as nominals for $w_1, \ldots, w_n$.

$$\delta_{\mathsf{F}}(p_1, \ldots, p_n) := \bigwedge_{i=1}^{n} \langle \mathsf{U} \rangle p_i \wedge \bigvee_{i=1}^{n} p_i \wedge \bigwedge_{1 \leq i \neq j \leq n} (p_i \to \neg p_j) \wedge$$

$$\bigwedge_{1 \leq i,j \leq n} \{p_i \to \Diamond p_j | w_i R w_j\} \wedge \bigwedge_{1 \leq i,j \leq n} \{p_i \to \neg \Diamond p_j | \neg w_i R w_j\}.$$

When $p_1, \ldots, p_n$ are fixed or known from the context, I will write simply χ_{F}.

A folklore fact

Lemma

For any frame G *and finite frame* F *the following are equivalent:*

1. $\mathsf{G} \twoheadrightarrow \mathsf{F}$.
2. $\mathsf{G} \not\models \chi_{\mathsf{F}}$.

The modal logics $\mathbf{ML}^{\mathsf{r}}$ and $\mathbf{ML}^{\mathsf{as}}$

The modal logics $\mathbf{ML}^{\mathsf{r}}$ and $\mathbf{ML}^{\mathsf{as}}$

$\mathbf{ML}^{\mathsf{r}}$: the modal logic of F^{r}.

$\mathbf{ML}^{\mathsf{as}}$: the modal logic of all formulae which are almost surely valid in finite frames.

Proposition (G. & Kapron, 2003)

1. $\mathbf{ML}^{\mathsf{r}}$ *and* $\mathbf{ML}^{\mathsf{as}}$ *are normal modal logics.*
2. *A modal formula* ϕ *is in* $\mathbf{ML}^{\mathsf{r}}$ *iff its frame condition* $FC(\phi)$ *follows from some extension axiom. Therefore, every such formula is in* $\mathbf{ML}^{\mathsf{as}}$.

   Consequently, $\mathbf{ML}^{\mathsf{r}} \subseteq \mathbf{ML}^{\mathsf{as}}$.

Are there any formulae in $\mathbf{ML}^{\mathsf{as}}$ that are not in $\mathbf{ML}^{\mathsf{r}}$?

Axiomatizing the logic $\mathbf{ML}^{\mathrm{r}}$

A complete axiomatization of $\mathbf{ML}^{\mathrm{r}}$ (G. & Kapron, 2003):

Axioms:

(ML_1^r) K: $\Box(p \to q) \to (\Box p \to \Box q)$.
(ML_2^r) $[\mathrm{U}]p \to p$.
(ML_3^r) $[\mathrm{U}]p \to [\mathrm{U}]\Box p$.
(ML_4^r) $p \to [\mathrm{U}]\langle \mathrm{U}\rangle p$.
(ML_5^r) Scheme MODEXT of the following axioms for each $n \in \mathbb{N}$:

$$\mathrm{MODEXT}_n = \bigwedge_{k=1}^{n} \langle \mathrm{U}\rangle(p_k \wedge \Box q_k) \to \langle \mathrm{U}\rangle \bigwedge_{k=1}^{n} (\Diamond p_k \wedge q_k).$$

Inference rules: MP and Necessitation.

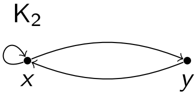
Frame condition of MODEXT_n: every set of n points has a common predecessor which is also their common successor. In all at most n-element frames this implies existence of a central point x, i.e. such that Rxy and Ryx hold for every $y \in W$.

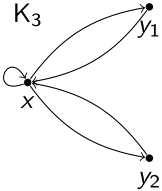
Theorem (G. & Kapron, 2003)

$\mathbf{ML}^{\mathrm{r}}$ *is not finitely axiomatizable, yet it has the finite model property and is decidable (in ExpTime).*

Two examples of finite frames with a central point

$K_2 = \langle \{x, y\}, \{(x, x), (x, y), (y, x)\} \rangle$
$K_3 = \langle \{x, y_1, y_2\}, \{(x, x), (x, y_1), (x, y_2), (y_1, x), (y_2, x)\} \rangle.$





The kernel partition $\mathcal{P}_{K_2}$ corresponds to the standard notion of kernel in digraphs.

The kernel partition $\mathcal{P}_{K_3}$: a "double kernel".

The finite frames of $\mathbf{ML}^{\mathsf{r}}$

Proposition (G. & Kapron, 2003)

For every finite frame F *the following are equivalent.*

1. $\mathsf{F} \models \mathbf{ML}^{\mathsf{r}}$
2. F *has a central point.*
3. F *is a bounded-morphic image of* F^{r}.
4. $\mathsf{F}^{\mathsf{r}} \not\models \chi_{\mathsf{F}}$.
5. F *can be obtained from* F^{r} *by filtration.*

Corollary

For every finite frame G *without central point:* $\chi_{\mathsf{G}} \in \mathbf{ML}^{\mathsf{r}}$.

Failure of the transfer theorem for modal frame validity

Corollary

F^{r} *has all kernel partitions defined by finite frames with a central point.*

In particular, χ_{K_2} and χ_{K_3} fail in F^{r} and F^{r} has a kernel and a double kernel.

But, is the existence of such kernels almost surely true in the finite? No!

Theorem (G. & Kapron, 2003)

The existence of a double kernel is almost surely false in finite frames.
Therefore, $\chi_{\mathsf{K}_3} \in \mathbf{ML}^{\mathsf{as}} \setminus \mathbf{ML}^{\mathsf{r}}$.

Proof: by combinatorial-analytic estimation of the expected number of double kernels in a random finite frame, which is proved to be asymptotically 0.

Same results were proved later for single kernels and χ_{K_2} by Le Bars (LICS'2002).

Corollary

1. *The transfer theorem for frame validity in modal logic fails.*
2. $\mathbf{ML}^{\mathsf{r}}$ *is strictly included in* $\mathbf{ML}^{\mathsf{as}}$.

Back to the modal logic of almost sure frame validity $\mathbf{ML}^{\mathsf{as}}$

What axioms must be added to $\mathbf{Ax}(\mathbf{ML}^{\mathsf{r}})$ to axiomatize completely $\mathbf{ML}^{\mathsf{as}}$?

Proposition (G. & Kapron, 2003)

1. *Every first-order definable modal formula which is in* $\mathbf{ML}^{\mathsf{as}}$ *is also in* $\mathbf{ML}^{\mathsf{r}}$.
2. *Every modal formula* ϕ *in* $\mathbf{ML}^{\mathsf{as}}$ *that defines a purely universal* (Π^1_1) *frame condition is valid, hence also in* $\mathbf{ML}^{\mathsf{r}}$.

So, the missing axioms are neither first-order definable, nor purely universal.

How to identify them?

The almost sure characteristic formulae

Proposition

For every finite frame F*:* $\mathsf{F} \models \mathbf{ML}^{\mathsf{as}}$ *iff* $\chi_{\mathsf{F}} \notin \mathbf{ML}^{\mathsf{as}}$.

Consequently, if $\mathsf{F} \not\models \phi$ *for some* $\phi \in \mathbf{ML}^{\mathsf{as}}$ *then* $\chi_{\mathsf{F}} \in \mathbf{ML}^{\mathsf{as}}$.

Towards axiomatizing the logic $\mathbf{ML}^{\mathsf{as}}$

So, natural candidates for additional axioms of $\mathbf{ML}^{\mathsf{as}}$ over $\mathbf{Ax}(\mathbf{ML}^{\mathsf{r}})$ are the almost surely valid formulae of the type χ_{F} for frames F with central point:

$$\Xi^{\mathsf{as}} := \{\chi_{\mathsf{F}} \mid \mathsf{F} \in \mathcal{C} \text{ and } \chi_{\mathsf{F}} \in \mathbf{ML}^{\mathsf{as}}\}.$$

where $\mathcal{C}$ is the set of all finite frames with a central point. (NB: $\mathcal{C} \subseteq \mathcal{F}^{\mathsf{d2}}$.)

Conjecture

$\mathbf{Ax}(\mathbf{ML}^{\mathsf{r}}) \cup \Xi^{\mathsf{as}}$ axiomatizes $\mathbf{ML}^{\mathsf{as}}$.

Notation: given a set of formulae Γ and a formula ϕ, $\Gamma \models^{\mathsf{fr}}_{\mathsf{fin}} \phi$ means that ϕ is valid in every finite frame in which all formulae of Γ are valid.

An encouraging observation in support of the conjecture:

Proposition

For any $\phi \in \mathbf{ML}^{\mathsf{as}}$: $\mathbf{Ax}(\mathbf{ML}^{\mathsf{r}}) \cup \Xi^{\mathsf{as}} \models^{\mathsf{fr}}_{\mathsf{fin}} \phi$.

Axiomatizing $\mathbf{ML}^{as}$: two major problems

Two major problems with proving the Conjecture, if true at all:

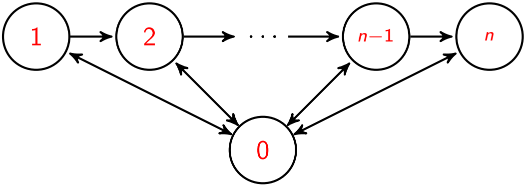
1. How to explicitly identify the axioms in Ξ^{as}?
2. How to prove the completeness?

How many additional axioms are needed for $\mathbf{ML}^{\mathrm{as}}$?

Proposition

There is a set $\Phi = \{\chi_{\mathsf{F}_n}\}_{n\in\mathbb{N}}$ *of infinitely many axioms in* Ξ^{as}*,*
none of which follows from all others in terms of $\models^{\mathrm{fr}}_{\mathrm{fin}}$ *(and, hence, deductively).*

Consider F_n:



Proof sketch: $\alpha = \forall x \exists y (Rxy \wedge \neg Ryy)$ is an instance of $(\mathrm{EXT})_1$ that fails in each F_n, hence in every G such that $\mathsf{G} \twoheadrightarrow \mathsf{F}$, as α is preserved in bounded morphic images.

Thus, the set of finite frames G such that $\mathsf{G} \not\twoheadrightarrow \mathsf{F}$ has a measure 1 in $\mathcal{F}^{\mathrm{fin}}$.

Therefore, each χ_{F_n} is in $\mathbf{ML}^{\mathrm{as}}$.

Now, let $\Phi^{-n} = \{\chi_{\mathsf{F}_m} \mid m > 0, m \neq n\}$.

Then $\Phi^{-n} \not\models^{\mathrm{fr}}_{\mathrm{fin}} \chi_{\mathsf{F}_n}$ for each $n \in \mathbb{N}$, because $\mathsf{F}_n \models \Phi^{-n}$, while $\mathsf{F}_n \not\models \chi_{\mathsf{F}_n}$.

Conjecture

The logic $\mathbf{ML}^{\mathrm{as}}$ *is not finitely axiomatizable over* $\mathbf{ML}^{\mathrm{r}}$*.*

Axiomatizing $\mathbf{ML}^{\mathsf{as}}$: further speculations

A partial syntactic description of the missing axioms in Ξ^{as} is given in the paper, using the extension axioms and van Benthem's characterisation the FO sentences in the language with $=$ and R that are preserved under bounded morphisms.

The big unknown: are these *all* axioms that are missing, or are there more, that are not identifiable in such a way?

- If these are all, then the logic $\mathbf{ML}^{\mathsf{as}}$ is recursively axiomatizable over $\mathbf{ML}^{\mathsf{r}}$ and even stands a chance to be decidable, too (like $\mathbf{ML}^{\mathsf{r}}$).
- Otherwise, the problem of identifying all missing axioms is very likely going beyond modal logic, into complex combinatorial-probabilistic calculations.

Axiomatizing $\mathbf{ML}^{\mathsf{as}}$: summing up

To sum up: it is currently unknown whether the set Ξ^{as} is recursive, or even recursively enumerable. (I conjecture that it is.)

But, even if that is the case, the question whether $\mathbf{ML}^{\mathsf{r}} \cup \Xi^{\mathsf{as}}$ axiomatizes $\mathbf{ML}^{\mathsf{as}}$ remains open.

The reason is that we cannot conclude $\mathbf{ML}^{\mathsf{r}} \cup \Xi^{\mathsf{as}} \vdash \phi$ from $\mathbf{ML}^{\mathsf{r}} \cup \Xi^{\mathsf{as}} \models^{\mathsf{fr}}_{\mathsf{fin}} \phi$, unless we have a recursive axiomatization of $\models^{\mathsf{fr}}_{\mathsf{fin}}$ in modal logic.
(I currently do not know if one exists, but suspect it does not.)

An important related question (raised by E. Zolin) is whether $\mathbf{ML}^{\mathsf{as}}$ is *Kripke complete*, i.e. whether it is the modal logic of *any* class of Kripke frames.

On proving completeness of the axiomatization of $\mathbf{ML}^{as}$

Even when the axiomatization is eventually obtained, the problem of proving its completeness seems not less challenging.

Indeed, unlike the axiom scheme MODEXT, the truly second-order axioms for $\mathbf{ML}^{as}$, like those from Ξ^{as}, are likely not to be canonical.

But, some recent results and techniques developed by N. and G. Bezhanishvili and others give some hope.

Still, how difficult that problem is may only be properly assessed when all axioms are explicitly known.

In summary: the question of establishing a provably complete axiomatization of $\mathbf{ML}^{as}$, while better understood now, remains open.

Further agenda

Many other related problems arise.

Just one such generic question: given a class $\mathcal{F}$ of Kripke frames, what is the modal logic of almost sure validities of $\mathcal{F}$?

The case when the modal logic of $\mathcal{F}$ satisfies the 0-1 law seems to be considerably easier (though, by no means trivial) than the case of $\mathcal{F} = \mathcal{F}^{\mathsf{fin}}$ studied here, as it then boils down to axiomatizing the modal logic of the respective analogue of countable random frame, relativised to the class $\mathcal{F}$, if it exists.

Promising recent results of that type were announced by R. Verbrugge (AiML'2018) for the modal provability logic and two versions of Grzegorczyk logic.

Going beyond modal logic, the problems of axiomatizing the almost sure validity in the finite for UMSO, EMSO, and the full MSO on graphs, digraphs, and other important classes of structures naturally arise.

These very likely lead to complicated combinatorial-probabilistic computations proving almost sure existence (resp., non-existence) of kernel partitions.

In general, little is known so far about these theories and the challenge to understand them is wide open.

Concluding remarks

- Asymptotic probabilities and 0-1 laws on properties of random finite structures are phenomena deeply relating logic, combinatorics and probability.
- While 0-1 laws can easily fail in extensions of FO, the almost sure validities in the finite in a given logical language over a given class of structures always forms a well defined logical theory.
- Axiomatizing such a logical theory may lead to quite complicated combinatorial-probabilistic computations.
- In the case of modal logic, that theory is conjectured to be axiomatized by the characteristic formulae of all finite frames creating kernel partitions whose existence in finite frames is almost surely false.
- The complete axiomatization of $\mathbf{ML}^{as}$ is still an open problem.
- Likewise respectively for UMSO and EMSO.
  In general, little is known about these so far. Challenges abound.
- Possible applications: to average case complexity and algorithmic optimization.

THE END