On some questions around Berest's conjecture

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Consider an irreducible polynomial $f(X,Y) \in \mathbb{C}[X,Y]$. Assume there exists a pair of operators $(P,Q) \in A_1$, where $A_1 = \mathbb{C}[x][\partial]$ is the first Weyl algebra, such that $f(P,Q) = 0$ and $\mathrm{ord}(P) > 0$ or $\mathrm{ord}(Q) > 0$, where $\mathrm{ord}(P) = k$ if $P = \sum_{i=0}^{k} a_i \partial^i$ with $a_i \neq 0$ (we'll call such a pair *a non-trivial solution* of the equation $f = 0$).

The group of automorphisms of the first Weyl algebra A_1 acts on the set of solutions of the equation $f(X,Y) = 0$, i.e. if $P, Q \in A_1$ satisfy the equation and $\varphi \in Aut(A_1)$, then $\varphi(P), \varphi(Q)$ also satisfy the equation. A natural and important problem is to describe the orbit space of the group action of $Aut(A_1)$ in the set of solutions.

Y. Berest, cf. [6], proposed the following interesting conjecture:

If the Riemann surface corresponding to the equation $f = 0$ has genus $g = 1$ then the orbit space is infinite, and if $g > 1$ then there are only finite number of orbits.

This conjecture was first studied in the work [6], where its connection with the well known open conjecture of Dixmier was announced. Recall that the Dixmier conjecture (formulated in his seminal work [2]) claims that any non-zero endomorphism of A_1 is actually an automorphism.

The Berest conjecture was studied in several papers: in [6] and [7] (cf. also [5]), in [1] and in [3].

In this talk we present two conclusions around Berest's conjecture:

Теорема 1. *Let $f \in K[X,Y]$, $f(X,Y) = \sum_{i+j<N} c_{ij} X^i Y^j$ be a non-zero polynomial. Assume that $f(P,Q) = \sum c_{ij} P^i Q^j = 0$ where $P, Q \in A_1$. Then $[P,Q] = 0$.*

The main result is the following theorem

Теорема 2. *Assume there exists a non-zero $f(X,Y) \in K[X,Y]$, which has a non-trivial solution $(P,Q) \in A_1^2$, and the number of orbits of its solutions in A_1^2 is finite. Then the Dixmier conjecture holds, i.e $\forall \varphi \in End(A_1) - \{0\}$, φ is an automorphism.*

The talk is based on the joint work [4] with A.B. Zheglov.

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