Symbolic Dynamics

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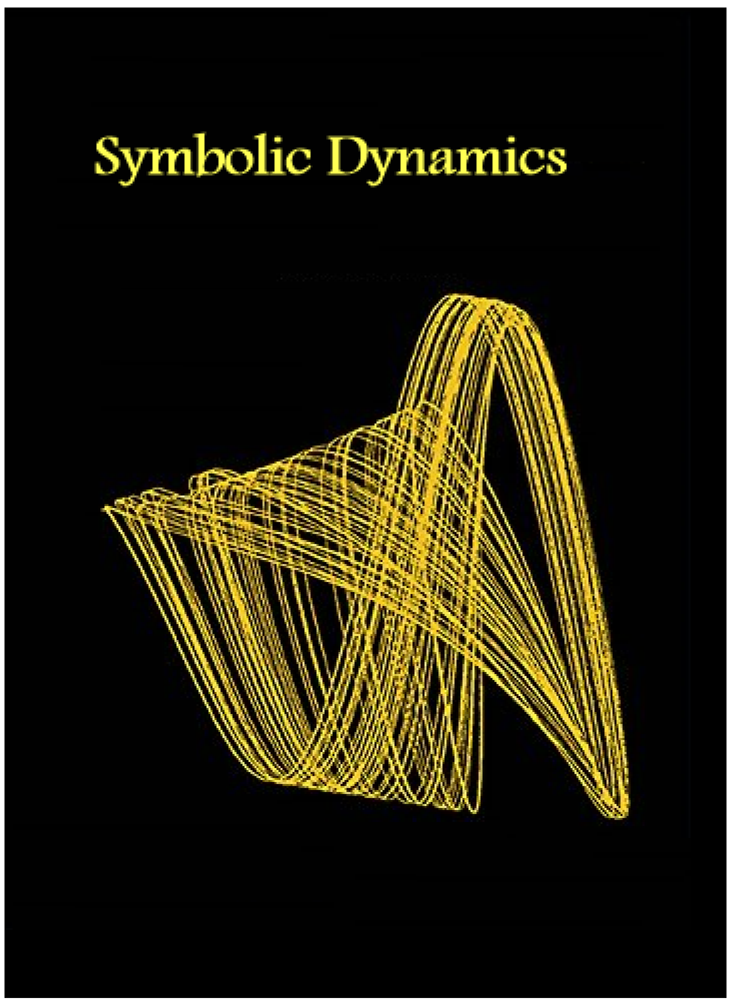

Course Description

This course is about one the branches of dynamical systems; What a dynamical systems is a space with a bunch of rules that on it that tell us what happens to each point in that space; Yet what we want to do is then be able to predict long-term behavior about what happens to the system as you keep applying rule over and over this kind of space the ones.

We look at every abstract but you could think of them as something like planets rotating around the sun and you take a snapshot where each one is once a second and then try to tell me where they will be millions of years into future.

One of the tools we can use in our research is **symbolic dynamics** and what this is a mechanism to take a complicated abstract dynamical system and record it in a simpler way so that we can study it.

Let's say you are standing and observing a traffic light. Then we are going to pretend there's no yellow to make it simpler; it is just alternates between red and green; and let's say it changes every 30 seconds, and every 30 seconds. We might down the color of the traffic light using a **0** to say that it's red, and a **1** to say that it's green. Then our pattern will just say **01010...**. It will continue like that forever more in the other words this is a super simple pattern, because it's just this little block **01** repeated; all the time it's a periodic pattern.

Now, let's look at that same traffic light, but we don't measure it every 30 seconds. Maybe we measure it every 31 seconds or 32 seconds. When we do this, we might see the same number in a row twice, so this is something that we can detect using symbolic dynamics.

It is really a beautiful part of math to find connections which are not apparent and finding structures; where you didn't know that, they were going to be there to us this is one of the best parts about doing.

Course Outline

- **Shift Spaces:** alphabet, words (finite, infinite, bi-infinite), full shifts, subwords, concatenated, shift map, stationary, periodicity, shift space, subshift, examples of shifts (subshift, even, run-length limited, S-gap, prime gap, charge constrained, finite type, context-free) shift map, language, irreducible shift, higher block code, higher block presentation, higher power shift, conjugate shift space
- **Shifts of Finite Type:** shift of finite type, memory-M of finite type, edge shift, vertex shif, a bi-infinite walk, transposed shift, Markov chain, state split graph, division matrix
- **Sofic Shifts:** symbolic adjacency matrix, sofic shift, follower set graph, synchronizing word
- **Entropy:** entropy, Perron eigenvalue, Perron eigenvector, period of a state, mixing shift spaces
- **Finite-State Codes:** road-coloring, finite-state code, sliding block decoder
- **Shifts and Dynamical Systems:** dynamical system, invertible dynamical system, examples(doubling map, toral automorphism, shift dynamical system, topological conjugacy, topological entropy, zeta function, topological partition, symbolic dynamical system, expanding and contracting segment, Intersection Property, horseshoe mapping

Theorems

Theorem 1: The image of a shift space under a sliding block code is a shift space.

Theorem 2: A sliding block code that is one-to-one and onto has a sliding block inverse, and is hence a conjugacy.

Theorem 3: A shift space X is an M-step shift of finite type if and only if whenever $uv, vw \in \mathcal{B}(X)$ and $|v| \geq M$, then $uvw \in \mathcal{B}(X)$.

Theorem 4: If X is an M-step shift of finite type, then there is a graph G such that $X^{[M+1]} = X_G$.

Theorem 5: If a graph H is a splitting of a graph G, then the edge shifts X_G and X_H are conjugate.

Theorem 6: Sofic shifts are shift spaces.

Theorem 7: Every shift of finite type is sofic.

Theorem 8: A shift space is sofic if and only if it is a factor of a shift of finite type.

Theorem 9: A shift space is sofic if and only if it has a finite number of follower sets.

Theorem 10: Every sofic shift has a right-resolving presentation.

Theorem 11: Any two minimal right-resolving presentations of an irreducible sofic shift are isomorphic as labeled graphs.

Theorem 12: There is an algorithm which, given two labeled graphs $\mathcal{G}_1$ and $\mathcal{G}_2$, decides whether $X_{\mathcal{G}_1} = X_{\mathcal{G}_2}$.

Theorem 13: Let G be a graph having adjacency matrix A. Then $h(X_G) = \log \lambda_A$.

Theorem 14: (The Road Theorem) Every primitive road-colorable graph has a road-coloring for which there is a synchronizing word.

Theorem 15: (Finite-State Coding Theorem).

Theorem 16: (Sliding Block Decoding Theorem).

Theorem 17: A subset of $\mathcal{A}^{\mathbb{Z}}$ is a shift space if and only if it is shift-invariant and compact.

Theorem 18: (Curtis–Hedlund–Lyndon Theorem).

Prerequisites/Corequisites

We will cover every necessary prerequisites.

Main References

1. D. Lind and B. Marcus, *An Introduction to Symbolic Dynamics and Coding*, Cambridge University Press, 2021.

2. B. Kitchens, *Symbolic Dynamics (One-sided, Two-sided and Countable State Markov Shifts)*, Springer, 1998.

3. A Ya Belov, Grigorii V Kondakov, Ivan Mitrofanov *Inverse problems of symbolic dynamics* , arXiv preprint arXiv:1104.5605

4. A. Ya. Kanel-Belov, G. V. Kondakov, I. V. Mitrofanov, M. M. Golafshan, 2021, *On the sequence of the first binary digits of the fractional parts of the values of a polynomial*, Chebyshevskii sbornik, vol. 22, no. 1, pp. 482–487.

Grading Policy

- Assignments; (80%)
- Projects. (20%)

Assignments

5 series of homework: each section one homework include of one or two problems

Project

Based on materials of class